

Density analysis of BSDEs

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Abstract

In this paper, we study the existence of densities (with respect to the Lebesgue measure) for marginal laws of the solution (Y, Z) to a quadratic growth BSDE. Using the (by now) well-established connection between these equations and their associated semi-linear PDEs, together with the Nourdin-Viens formula, we provide estimates on these densities.

Key words: BSDEs; Malliavin Calculus; Density analysis; Nourdin-Viens' Formula; PDEs.

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1 Introduction

In recent years the field of Backward Stochastic Differential Equations (BSDEs) has been a subject of growing interest in stochastic calculus, as these equations naturally arise in stochastic control problems in Finance, and as they provide Feynman-Kac type formulae for semi-linear PDEs ([24]). Before going further let us recall that a solution to a BSDE is a pair of *regular enough* (in a sense to be made precise) predictable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T h(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (1.1)$$

where W is a one-dimensional Brownian motion, h is a predictable process and ξ is a \mathcal{F}_T -measurable random variable (with $(\mathcal{F}_t)_{t \in [0, T]}$ the natural completed and right-continuous filtration generated by W). Since it is generally not possible to provide an explicit solution to (1.1), except for instance when h is a linear mapping of (y, z) , one of the main issues especially regarding the applications is to provide a numerical analysis for the solution of a BSDE. This calls for a deep understanding of the regularity of the solution processes Y and Z . The classical regularity related to the obtaining of a numerical scheme for the solution (Y, Z) is the so-called *path regularity* for the Z component

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originally studied in [18]. In this paper we aim at studying another type of regularity namely, we focus on the law of the marginals of the random variables Y_t, Z_t at a given time t in $(0, T)$. More precisely, we are interested in providing sufficient conditions which ensure the existence of a density (with respect to the Lebesgue measure) for these marginals on the one hand, and in deriving some estimates on these densities on the other hand. This type of information on the solution is of theoretical and of practical interest since the description of the tails of the (possible) density of Z_t would provide more accurate estimates on the convergence rates of numerical schemes for quadratic growth BSDEs (qgBSDEs in short), that is when h in (1.1) has quadratic growth in the z -variable, as noted in [6].

Coming back to the main problem under interest of this paper, i.e. existence of densities for the marginal laws of Y and Z , it is worth mentioning that this issue has been pretty few studied in the literature, since up to our knowledge only references [3, 1] address this question. The first results about this problem have been derived in [3], where the authors provide existence and smoothness properties of densities for the marginals of the Y component only and when the driver h is Lipschitz continuous in (y, z) . Note that two kinds of sufficient conditions for the existence of a density for Y are derived in [3]: the so-called *first-order* (cf. [3, Theorem 3.1]) and *second-order* (see [3, Theorem 3.6]) conditions. Concerning the Z component, much less is known since existence of a density for Z has been established in [1] only under the condition that the driver is linear in z . This constitutes, to our point of view, a major restriction since up to a Girsanov transformation this case basically reduces to the case where the driver does not depend on z . Nonetheless, in [1], estimates on the densities of the laws of Y_t and Z_t are given using the Nourdin-Viens formula.

In this paper we revisit and extend the results of [3, 1] by providing sufficient conditions for the existence of densities for the marginal laws of the solution Y_t, Z_t (with t an arbitrary time in $(0, T)$) of a qgBSDE with a terminal condition ξ in (1.1) given as a deterministic mapping of the value at time T of the solution to a one-dimensional SDE, together with some estimates on these densities. The results concerning the Y component are presented in Section 3. As recalled above, the case where h is Lipschitz continuous in (y, z) has been investigated in [3] where the authors have derived two types of sufficient conditions. However, we provide as Example 3.5 a counter-example to [3, Theorem 3.6] which is devoted to the second-order conditions. In addition, we propose a new version of this result as Theorem 3.6. Then, we gather in Section 3.2 the first existence results of a density for the Y component for qgBSDEs. Concerning the Z component we propose in Section 4 sufficient conditions for the existence of a density first for Lipschitz BSDEs (in Section 4.1), then for qgBSDEs (see Section 4.2). We would like to stress once more at this stage that only the case of linear drivers in z was known (see [1, Theorem 4.3]) up to now, which makes our result a major improvement on the existing literature. Finally, we derive in Section 5, density estimates for the marginal laws of Y and Z using the Nourdin-Viens formula, and taking advantage of the connection between the solution to a Markovian BSDE and the solution to its associated semi-linear PDE. Note that contrary to [1], we do not assume that the Malliavin derivative of Y (or Z) to be bounded which is, from our point of view, a too stringent assumption (as illustrated in Example 5.1) both from the theoretical and practical point of view. Indeed, such an assumption leads to Gaussian tails for the densities of Y or Z . However, even in seemingly benign situations, we will see that it is not generally the case for BSDEs, and unlike most of the literature, we have obtained tail estimates which are *not* Gaussian. This might be seen as a significant difference between BSDEs and diffusive equations (i.e. with an initial condition) like SDEs or SPDEs for instance [17, 16, 22].

2 Preliminaries

2.1 General notations

In this paper we fix $T \in (0, \infty)$. Let $W := (W_t)_{t \in [0, T]}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ the natural (completed and right-continuous) filtration generated by W . We denote by λ the Lebesgue measure on \mathbb{R} and we set for any $p \in [1, +\infty]$, $L^p(\mathbb{P}) := L^p(\Omega, \mathcal{F}_T, \mathbb{P})$ and denote by $\|\cdot\|_p$ the associated norm. We denote by $\mathcal{C}_b(\mathbb{R}^n)$ ($n \geq 1$) the set of functions from \mathbb{R}^n to \mathbb{R} which are infinitely differentiable with bounded partial derivatives. Similarly, for any $n \geq 1$ and any $p \in \mathbb{N}^*$, we denote by $\mathcal{C}^p(\mathbb{R}^n)$ the set of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which are p -times continuously differentiable. For f in $\mathcal{C}_b(\mathbb{R}^n)$ we set $f_{x_{i_1} \dots x_{i_n}}$ the n -th partial derivative with respect to the variables x_{i_1}, \dots, x_{i_n} with $i_1 + \dots + i_n = n$. For a differentiable mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, we denote f' its derivative in place of f_x . Let us denote, for any $(p, q) \in \mathbb{N}^2$, by $\mathcal{C}^{p,q}$ the space of functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ which are p -times differentiable in t and q -times differentiable in space with partial derivatives continuous (in (t, x)).

Finally, we introduce the following norms and spaces for any $p \geq 1$. \mathbb{S}^p is the space of \mathbb{R} -valued, continuous and \mathbb{F} -progressively measurable processes Y s.t.

$$\|Y\|_{\mathbb{S}^p}^p := \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty.$$

\mathbb{S}^∞ is the space of \mathbb{R} -valued, continuous and \mathbb{F} -progressively measurable processes Y s.t.

$$\|Y\|_{\mathbb{S}^\infty} := \sup_{0 \leq t \leq T} \|Y_t\|_\infty < +\infty.$$

\mathbb{H}^p is the space of \mathbb{R} -valued and \mathbb{F} -predictable processes Z such that

$$\|Z\|_{\mathbb{H}^p}^p := \mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty.$$

BMO is the space of square integrable, continuous, \mathbb{R} -valued martingales M such that

$$\|M\|_{\text{BMO}} := \text{ess sup}_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}_\tau \left[(M_T - M_{\tau-})^2 \right] \right\|_\infty < +\infty,$$

where for any $t \in [0, T]$, \mathcal{T}_t^T is the set of \mathbb{F} -stopping times taking their values in $[t, T]$. Accordingly, $\mathbb{H}_{\text{BMO}}^2$ is the space of \mathbb{R} -valued and \mathbb{F} -predictable processes Z such that

$$\|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 := \left\| \int_0^\cdot Z_s dB_s \right\|_{\text{BMO}}^2 < +\infty.$$

2.2 FBSDE and assumptions

In this paper, we consider a FBSDE of the form:

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, & t \in [0, T], \mathbb{P} - a.s. \\ Y_t = g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, & t \in [0, T], \mathbb{P} - a.s. \end{cases} \quad (2.1)$$

We denote by $\mathfrak{S}(X_t)$ the support of the law of X_t under \mathbb{P} , that is to say the smallest closed subset A of \mathbb{R} such that $\mathbb{P}(X_T \in A) = 1$. We list below all the different assumptions that will be of use in this paper.

List of assumptions:

- (A) (i) $b, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous in time and continuously differentiable in space for any fixed time t and such that there exist $k_b, k_\sigma > 0$ with

$$|b_x(t, x)| \leq k_b, \quad |\sigma_x(t, x)| \leq k_\sigma, \quad \text{for all } x \in \mathbb{R}.$$

Besides $b(t, 0), \sigma(t, 0)$ are bounded functions of t and there exists $c > 0$ such that for all $t \in [0, T]$

$$0 < c \leq |\sigma(t, \cdot)|, \quad \lambda(dx) - a.e.$$

- (ii+) For any $t \in [0, T]$, the maps $x \mapsto b(t, x)$ and $x \mapsto \sigma(t, x)$ are respectively in $\mathcal{C}^2(\mathbb{R})$ and $\mathcal{C}^3(\mathbb{R})$, and there exists $c > 0$ such that

$$\sigma \geq c > 0, \quad \sigma' \geq 0, \quad \sigma'', \sigma''' \leq 0 \quad \text{and} \quad [\sigma, [\sigma, b]] \geq 0,$$

where $[b, \sigma]$ denotes the Lie bracket between b and σ defined by $[b, \sigma] := b'\sigma + \sigma'b$.

- (ii-) For any $t \in [0, T]$, the maps $x \mapsto b(t, x)$ and $x \mapsto \sigma(t, x)$ are respectively in $\mathcal{C}^2(\mathbb{R})$ and $\mathcal{C}^3(\mathbb{R})$, and there exists $c < 0$ such that

$$\sigma \leq c < 0, \quad \sigma' \leq 0, \quad \sigma'', \sigma''' \geq 0 \quad \text{and} \quad [\sigma, [\sigma, b]] \leq 0.$$

Remark 2.1. *Let us comment on the above assumption. First of all, according to Theorem 2.1 in [10], (A)(i) implies that for all $t \in (0, T]$, the law of X_t , denoted by $\mathcal{L}(X_t)$, has a density with respect to the Lebesgue measure. Moreover, according to [1], (A)(ii+) and (A)(ii-) are sufficient conditions to ensure that D^2X exists and has a sign (see Remark 4.6 below for more details).*

- (L) (i) $g : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\mathbb{E}[g(X_T)^2] < +\infty$.
(ii) $h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is such that there exist $(k_x, k_y, k_z) \in (\mathbb{R}_+^*)^3$ such that for all $(t, x_1, x_2, y_1, y_2, z_1, z_2) \in [0, T] \times \mathbb{R}^6$,

$$|h(t, x_1, y_1, z_1) - h(t, x_2, y_2, z_2)| \leq k_x|x_1 - x_2| + k_y|y_1 - y_2| + k_z|z_1 - z_2|.$$

- (iii) $\int_0^T |h(s, 0, 0, 0)|^2 ds < +\infty$.

- (Q) (i) $g : \mathbb{R} \rightarrow \mathbb{R}$ is bounded.

- (ii) $h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is such that:

▷ There exists $(K, K_z, K_y) \in (\mathbb{R}_+^*)^3$ such that for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^3$

$$|h(t, x, y, z)| \leq K(1 + |y| + |z|^2), \quad |h_z|(t, x, y, z) \leq K_z(1 + |z|), \quad |h_y|(t, x, y, z) \leq K_y.$$

▷ There exists $C > 0$ such that for all $(t, x, y, z_1, z_2) \in [0, T] \times \mathbb{R}^4$

$$|h(t, x, y, z_1) - h(t, x, y, z_2)| \leq C(1 + |z_1| + |z_2|)|z_1 - z_2|. \quad (2.2)$$

- (iii) $\int_0^T |h(s, 0, 0, 0)|^2 ds < +\infty$.

Remark 2.2. Assumptions (L) and (Q) are well-known in the BSDE literature, and are sufficient to ensure existence and uniqueness of a solution to the FBSDE (2.1) respectively in the so-called Lipschitz (see [23]) and Quadratic (see [15]) frameworks.

- (D1) (i) g is differentiable, $\mathcal{L}(X_T)$ -a.e., g and g' have polynomial growth.
(ii) $(x, y, z) \mapsto h(t, x, y, z)$ is continuously differentiable for every t in $[0, T]$.
- (D2) (i) g is twice differentiable, $\mathcal{L}(X_T)$ -a.e., g , g' and g'' have polynomial growth.
(ii) $(x, y, z) \mapsto h(t, x, y, z)$ is twice continuously differentiable for every t in $[0, T]$.
- (C+) $h_x, h_{xx}, h_{yy}, h_{zz}, h_{xy} \geq 0$ and $h_{xz} = h_{yz} = 0$,
- (C-) $h_x, h_{xx}, h_{yy}, h_{zz}, h_{xy} \leq 0$ and $h_{xz} = h_{yz} = 0$.
- (M) There exists a function $f \in \mathcal{C}^2(\mathbb{R})$ such that for all $t \in [0, T]$: $X_t = f(t, W_t)$.

Remark 2.3. Assumptions (D1) and (D2) are linked to the existence of first and second-order Malliavin derivatives for the Y component of the solution of (2.1). We would like to point out to the reader that we only require the differentiability of g , $\mathcal{L}(X_T)$ -a.e.. Such a relaxation will be particularly useful in the quadratic case (i.e. when Assumption (Q) holds). We emphasize that when we work under Assumption (A), the law of X_T is absolutely continuous with respect to the Lebesgue measure and X_T has finite moments of any order. Thus thanks to standard approximation arguments, we can show that the usual chain rule formula of Malliavin calculus (see Proposition 1.2.3. in [21]) still holds for the random variable $g(X_T)$, under Assumptions (D1) or (D2).

Moreover, if we set for a fixed $s \in [0, T]$:

$$\mathfrak{S}_s := \bigcup_{r \in [s, T] \cap \mathbb{Q}} \mathfrak{S}(X_r),$$

one could assume that h is differentiable with respect to x almost everywhere on \mathfrak{S}_s . However, for the sake of simplicity, we have decided to refrain from doing so, even though our results carry on directly to this more general setting.

2.3 Elements of Malliavin calculus and density analysis

In this section we introduce the basic material on the Malliavin calculus that we will use in this paper. Set $\mathfrak{H} := L^2([0, T], \mathcal{B}([0, T]), \lambda)$, where $\mathcal{B}([0, T])$ is the Borel σ -algebra on $[0, T]$, and let us consider the following inner product on \mathfrak{H}

$$\langle f, g \rangle := \int_0^T f(t)g(t)dt, \quad \forall (f, g) \in \mathfrak{H}^2,$$

with associated norm $\|\cdot\|_{\mathfrak{H}}$. Let \mathcal{S} be the set of cylindrical functionals, that is the set of random variables F in $L^2(\mathbb{P})$ of the form

$$F = f(W_{t_1}, \dots, W_{t_n}), \quad (t_1, \dots, t_n) \in [0, T]^n, \quad f \in \mathcal{C}_b(\mathbb{R}^n), \quad n \geq 1. \quad (2.3)$$

For any F in \mathcal{S} of the form (2.3), the Malliavin derivative DF of F is defined as the following \mathfrak{H} -valued random variable:

$$DF := \sum_{i=1}^n f_{x_i}(W_{t_1}, \dots, W_{t_n}) \mathbf{1}_{[0, t_i]}. \quad (2.4)$$

It is then customary to identify DF with the stochastic process $(D_t F)_{t \in [0, T]}$. Denote then by $\mathbb{D}^{1,2}$ the closure of \mathcal{S} with respect to the Sobolev norm $\|\cdot\|_{1,2}$, defined as:

$$\|F\|_{1,2} := \mathbb{E} [|F|^2] + \mathbb{E} \left[\int_0^T |D_t F|^2 dt \right].$$

In an iterative way, one may define $D^n F$ (for $n \geq 1$) as the following $\mathfrak{H}^{\odot n}$ -valued random variable:

$$D^n F := D(D^{n-1} F),$$

where $\mathfrak{H}^{\odot n}$ denotes the n -times symmetric tensor product of \mathfrak{H} . We refer to [21] for more details. We recall the following criterion for absolute continuity of the law of a random variable F with respect to the Lebesgue measure.

Theorem 2.4 (Bouleau-Hirsch, see e.g. Theorem 2.1.2 in [21]). *Let F in $\mathbb{D}^{1,2}$. Assume that $\|DF\|_{\mathfrak{H}} > 0$, \mathbb{P} -a.s. Then F has a probability distribution which is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .*

Let F such that $\|DF\|_{\mathfrak{H}} > 0$, \mathbb{P} -a.s., then the previous criterion implies that F admits a density ρ_F with respect to the Lebesgue measure. Assume there exists in addition a measurable mapping Φ_F with $\Phi_F : \mathbb{R}^{\mathfrak{H}} \rightarrow \mathfrak{H}$, such that $DF = \Phi_F(W)$, then we set:

$$g_F(x) := \int_0^\infty e^{-u} \mathbb{E} \left[\mathbb{E}^* [\langle \Phi_F(W), \widetilde{\Phi}_F^u(W) \rangle_{\mathfrak{H}}] \middle| F - \mathbb{E}(F) = x \right] du, \quad x \in \mathbb{R}, \quad (2.5)$$

where $\widetilde{\Phi}_F^u(W) := \Phi_F(e^{-u}W + \sqrt{1 - e^{-2u}}W^*)$ with W^* an independent copy of W defined on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, and \mathbb{E}^* denotes the expectation under \mathbb{P}^* (Φ_F being extended on $\Omega \times \Omega^*$). We recall the following result from [20].

Theorem 2.5 (Nourdin-Viens' formula). *F has a density ρ with the respect to the Lebesgue measure if and only if the random variable $g_F(F - \mathbb{E}[F])$ is positive a.s.. In this case, the support of ρ , denoted by $\text{supp}(\rho)$, is a closed interval of \mathbb{R} and for all $x \in \text{supp}(\rho)$:*

$$\rho(x) = \frac{\mathbb{E}(|F - \mathbb{E}[F]|)}{2g_F(x - \mathbb{E}[F])} \exp \left(- \int_0^{x - \mathbb{E}[F]} \frac{udu}{g_F(u)} \right).$$

3 Existence of a density for the Y component of a BSDE

In this section, we focus on the Y component of the FBSDE (2.1). The problem of existence of a density for the marginal laws of Y has been first studied in [3], when the generator h is assumed to be uniformly Lipschitz continuous in y and z . We first recall in Section 3.1 the results from [3]. Then, we point out a flaw in [3, Theorem 3.6] by providing a counter example to this result, and we propose a corrected version of it as Theorem 3.6. Next, in Section 3.2, we study the existence of a density for the marginal laws of Y when the generator h of the BSDE is quadratic in z .

We start by recalling the following (by now) classical result about existence and uniqueness of BSDEs (we refer to [23, 9, 15]).

Proposition 3.1. *(Existence and uniqueness of BSDEs) Under Assumptions (A)(i) and (L) or (Q), there exists a unique solution (X, Y, Z) in $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2$ to the FBSDE (2.1). In addition, when Assumption (Q) holds, we have $(Y, Z) \in \mathbb{S}^\infty \times \mathbb{H}_{\text{BMO}}^2$.*

Note that Condition (2.2) on the generator h in Assumption (Q) only ensures uniqueness of the solution. Hence, it can be dropped and one can then consider the maximal solution Y of the BSDE, for which our proofs still apply.

We now turn to the Malliavin differentiability of the processes (X, Y, Z) (see [23, 2] and [9, Remark of Proposition 5.3]).

Proposition 3.2. (*Malliavin differentiability*) Under (A)(i), (L) or (Q) and (D1), we have for any $t \in [0, T]$ that $(X_t, Y_t) \in (\mathbb{D}^{1,2})^2$, $Z_t \in \mathbb{D}^{1,2}$ for almost every t , and for all $0 < r \leq t \leq T$:

$$\begin{cases} D_r X_t = \sigma(r, X_r) + \int_r^t b_x(s, X_s) D_r X_s ds + \int_r^t \sigma_x(s, X_s) D_r X_s dW_s \\ D_r Y_t = g'(X_T) D_r X_T + \int_t^T H(s, D_r X_s, D_r Y_s, D_r Z_s) ds - \int_t^T D_r Z_s dW_s, \end{cases} \quad (3.1)$$

where $H(s, x, y, z) := h_x(s, X_s, Y_s, Z_s)x + h_y(s, X_s, Y_s, Z_s)y + h_z(s, X_s, Y_s, Z_s)z$.

Notice that BSDE (3.1) is a linear BSDE, whose solution can be computed using the linearization method (see [9]).

3.1 The Lipschitz case

We focus in this section on the existence of a density for the marginal laws of the process Y in the Lipschitz case, pursuing the study started in [3]. Towards this goal, we recall first the so-called *first order conditions* introduced in [3], which are only sufficient, as illustrated in Example 3.5. We then turn our attention to the *second-order conditions* of Theorem 3.6 in [3]. We point out a flaw in [3, Theorem 3.6] and provide a corrected version of this result as Theorem 3.6.

As in [3], we set for any $A \in \mathcal{B}(\mathbb{R})$ (i.e. the Borel σ -algebra on \mathbb{R}), and t in $[0, T]$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$:

$$\underline{g} := \inf_{x \in \mathbb{R}} g'(x), \quad \underline{g}^A := \inf_{x \in A} g'(x), \quad \bar{g} := \sup_{x \in \mathbb{R}} g'(x), \quad \bar{g}^A := \sup_{x \in A} g'(x), \quad (3.2)$$

$$\underline{h}(t) := \inf_{s \in [t, T], (x, y, z) \in \mathbb{R}^3} h_x(s, x, y, z), \quad \bar{h}(t) := \sup_{s \in [t, T], (x, y, z) \in \mathbb{R}^3} h_x(s, x, y, z). \quad (3.3)$$

Theorem 3.3. (*First-order conditions [3, Theorem 3.1]*) Assume that (A)(i), (L) and (D1) hold. Fix some $t \in (0, T]$ and set $K := k_b + k_y + k_\sigma k_z$. If there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and one of the two following assumptions holds

$$\begin{aligned} (H+) \quad & \begin{cases} \underline{g} e^{-\text{sgn}(\underline{g})KT} + \underline{h}(t) \int_t^T e^{-\text{sgn}(\underline{h}(s))Ks} ds \geq 0 \\ \underline{g}^A e^{-\text{sgn}(\underline{g}^A)KT} + \underline{h}(t) \int_t^T e^{-\text{sgn}(\underline{h}(s))Ks} ds > 0 \end{cases} \\ (H-) \quad & \begin{cases} \bar{g} e^{-\text{sgn}(\bar{g})KT} + \bar{h}(t) \int_t^T e^{-\text{sgn}(\bar{h}(s))Ks} ds \leq 0 \\ \bar{g}^A e^{-\text{sgn}(\bar{g}^A)KT} + \bar{h}(t) \int_t^T e^{-\text{sgn}(\bar{h}(s))Ks} ds < 0, \end{cases} \end{aligned}$$

then Y_t has a law absolutely continuous with respect to the Lebesgue measure.

Remark 3.4. Notice that \underline{g} (resp. \bar{g}) could be equal to $-\infty$ (resp. $+\infty$). Then Assumption (H+) (resp. (H-)) cannot be satisfied. Therefore, there is no problem if we allow the extrema of g to take the values $\pm\infty$.

Note that neither Condition (H+) nor Condition (H-) are necessary for getting existence of a density as illustrated in the following example.

Example 3.5. Let $T = 1$, $g(x) = x$, $X = W$, $h(s, x, y, z) = (s - 2)x$. In this case, $K = 0$ and $h_x(s, x, y, z) = s - 2$ for all $(x, y, z) \in \mathbb{R}^3$. For any t in $(0, 1]$, we have:

$$\bar{g} = \underline{g} = 1, \quad \underline{h}(t) = t - 2, \quad \bar{h}(t) = -1,$$

so that Assumption (H-) is not satisfied. Indeed,

$$\bar{g}e^{-\text{sgn}(\bar{g})KT} + \bar{h}(t) \int_t^T e^{-\text{sgn}(\bar{h}(s))Ks} ds = 1 - (1 - t) = t > 0.$$

Similarly, (H+) is not satisfied for any $t \in (0, (3 - \sqrt{5})/2)$ since:

$$\underline{g}e^{-\text{sgn}(\underline{g})KT} + \underline{h}(t) \int_t^T e^{-\text{sgn}(\underline{h}(s))Ks} ds = 1 + (t - 2)(1 - t) = -t^2 + 3t - 1,$$

which is negative for $t \in (0, (3 - \sqrt{5})/2)$. We deduce that for $t \in (0, (3 - \sqrt{5})/2)$ neither Assumption (H+) nor Assumption (H-) is satisfied. However, we know that:

$$\begin{aligned} Y_t &= \mathbb{E} \left[W_1 + \int_t^1 (s - 2)W_s ds \middle| \mathcal{F}_t \right] \\ &= W_t \left(1 + \int_t^1 (s - 2)ds \right) = W_t \left(-\frac{1}{2} + 2t - \frac{t^2}{2} \right), \quad \forall t \in [0, 1], \quad \mathbb{P} - a.s., \end{aligned} \quad (3.4)$$

which admits a density with respect to the Lebesgue measure except when $t = 0$ and $t = 2 - \sqrt{3}$.

Notice that in the previous example, the generator does not depend on z . In that setting, another result is derived [3] involving so-called *second order conditions*. However, Example 3.5 provides a counter-example to [3, Theorem 3.6]. Indeed, the second-order conditions proposed in [3, Theorem 3.6] entails that Y_t admits a density, when $t \neq \frac{1}{2}$, so in particular at $t = 2 - \sqrt{3}$. However from (3.4), $Y_{2-\sqrt{3}} = 0$. This example proves that [3, Theorem 3.6] has to be modified. We refer the reader to Example 3.7 below for more details and we propose a corrected version of [3, Theorem 3.6], in which the modified second-order conditions are sufficient, and necessary in the special situation of Example 3.5.

Consider the FBSDE (2.1) when h does not depend on z and define:

$$\begin{aligned} \tilde{h}(s, x, y, z) &:= - \left(h_{xt} + bh_{xx} - hh_{xy} + \frac{1}{2}(\sigma^2 h_{xxx} + 2z\sigma h_{xxy} + z^2 h_{xyy}) \right) (s, x, y) \\ &\quad - ((h_y + b_x)h_x + \sigma\sigma_x h_{xx} + z\sigma_x h_{xy}) (s, x, y). \end{aligned}$$

$$\tilde{g}(x) := g'(x) + (T - t)h_x(T, x, g(x)),$$

$$\underline{\tilde{g}} := \min_{x \in \mathbb{R}} \tilde{g}(x), \quad \bar{\tilde{g}} := \max_{x \in \mathbb{R}} \tilde{g}(x), \quad \underline{\tilde{g}}^A := \min_{x \in A} \tilde{g}(x), \quad \bar{\tilde{g}}^A := \max_{x \in A} \tilde{g}(x),$$

$$\underline{\tilde{h}}(t) := \min_{[t, T] \times \mathbb{R}^3} \tilde{h}(s, x, y, z), \quad \bar{\tilde{h}}(t) := \max_{[t, T] \times \mathbb{R}^3} \tilde{h}(s, x, y, z).$$

The following theorem corrects Theorem 3.6 in [3].

Theorem 3.6. Fix some $t \in (0, T]$, assume that h does not depend on z , that Assumptions (A)(i), (L) and (D1) hold and set $K := k_y + k_b$. If there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and one of the two following assumptions holds

$$\begin{aligned} \widetilde{(H+)} \quad & \begin{cases} \underline{\tilde{g}} e^{-\text{sgn}(\underline{\tilde{g}})KT} + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds \geq 0 \\ \underline{\tilde{g}}^A e^{-\text{sgn}(\underline{\tilde{g}}^A)KT} + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds > 0, \end{cases} \\ \widetilde{(H-)} \quad & \begin{cases} \overline{\tilde{g}} e^{-\text{sgn}(\overline{\tilde{g}})KT} + \overline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\overline{\tilde{h}}(s))Ks} (T-s) ds \leq 0 \\ \overline{\tilde{g}}^A e^{-\text{sgn}(\overline{\tilde{g}}^A)KT} + \overline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\overline{\tilde{h}}(s))Ks} (T-s) ds < 0, \end{cases} \end{aligned}$$

then the first component Y_t of the solution of BSDE (2.1) has a law which is absolutely continuous with respect to the Lebesgue measure.

Proof. We reproduce the proof of [3, Theorem 3.6] and provide a correction to their argument when necessary. We show that $\|DY_t\|_{\mathcal{H}} > 0$, \mathbb{P} -a.s.

Denote for notational simplicity $\Theta := (X, Y)$. For $0 \leq r \leq t \leq T$, $D_r Y_t$ writes down as:

$$D_r Y_t = g'(X_T) D_r X_T + \int_t^T h_x(s, \Theta_s) D_r X_s + h_y(s, \Theta_s) D_r Y_s ds + \int_t^T D_r Z_s dW_s. \quad (3.5)$$

Let $\gamma_{Y_t} := \int_0^T |D_r Y_t|^2 dr$. From (3.5), we deduce that

$$\gamma_{Y_t} = \left(\mathbb{E} \left[\psi_T g'(X_T) \zeta_T + \int_t^T \psi_s h_x(s, \Theta_s) \zeta_s ds \middle| \mathcal{F}_t \right] \right)^2 (\psi_t^{-1})^2 \int_0^t (\zeta_r^{-1} \sigma(r, X_r))^2 dr,$$

where

$$\psi_t := e^{\int_0^t h_y(u, \Theta_u) du}, \quad \zeta_t := e^{\int_0^t \sigma_x(s, X_s) dW_s + \int_0^t b_x(s, X_s) - \frac{1}{2} |\sigma_x(s, X_s)|^2 ds}.$$

The product $\psi_t \zeta_t$ can be rewritten as follows

$$\psi_t \zeta_t = \underbrace{\exp \left(\int_0^t b_x(s, X_s) + h_y(s, \Theta_s) ds \right)}_{=: E_t} \underbrace{\exp \left(\int_0^t \sigma_x(s, X_s) dW_s - \frac{1}{2} \int_0^t |\sigma_x(s, X_s)|^2 ds \right)}_{=: M_t},$$

from which we obtain

$$\gamma_{Y_t} = \left(\mathbb{E} \left[E_T M_T g'(X_T) + \int_t^T E_s M_s h_x(s, \Theta_s) ds \middle| \mathcal{F}_t \right] \right)^2 (\psi_t^{-1})^2 \int_0^t (\zeta_r^{-1} \sigma(r, X_r))^2 dr. \quad (3.6)$$

Let \mathbb{Q} be the probability measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} := M_T$, which is a positive martingale since σ_x is assumed to be bounded. By Girsanov's theorem, the stochastic process $W' := W - \int_0^\cdot \sigma_x(s, X_s) ds$ is a Brownian motion under \mathbb{Q} . Therefore, it is enough to show that under $\widetilde{(H+)}$

$$\mathbb{E}^{\mathbb{Q}} \left[E_T g'(X_T) + \int_t^T E_s h_x(s, \Theta_s) ds \middle| \mathcal{F}_t \right] \neq 0, \quad \mathbb{P} - \text{a.s.},$$

in order to obtain $\gamma_{Y_t} > 0$, \mathbb{P} -a.s.. Once more, we follow the original proof of [3, Theorem 3.6],

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} \left[E_T g'(X_T) + \int_t^T E_s h_x(s, \Theta_s) ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[E_T (g'(X_T) + h_x(T, \Theta_T)(T-t)) + \int_t^T (E_s h_x(s, \Theta_s) - E_T h_x(T, \Theta_T)) ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Integration by parts formula applied to $Eh_x(\cdot, \Theta)$ in the second summand above leads to

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} \left[E_T g'(X_T) + \int_t^T E_s h_x(s, \Theta_s) ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[E_T (g'(X_T) + h_x(T, \Theta_T)(T-t)) - \int_t^T \int_s^T E_r (h_{xt} + bh_{xx} - hh_{xy})(r, \Theta_r) dr ds \right. \\ & \quad - \int_t^T \int_s^T E_r \left(\sigma^2 \frac{1}{2} h_{xxx} + \sigma Z h_{xxy} + \frac{1}{2} Z^2 h_{xyy} + (b_x + h_y) h_x \right) (r, \Theta_r) dr ds \\ & \quad \left. - \int_t^T \int_s^T E_r (\sigma h_{xx} + Z h_{xy})(r, X_r, Y_r) dW_r ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

However, note that under \mathbb{Q} , W is not a Brownian motion and one needs to express the stochastic integrals in terms of W' . We obtain

$$\mathbb{E}^{\mathbb{Q}} \left[E_T g'(X_T) + \int_t^T E_s h_x(s, \Theta_s) ds \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[E_T \tilde{g}(X_T) + \int_t^T \int_s^T E_r \tilde{h}(r, \Theta_r) dr ds \middle| \mathcal{F}_t \right]. \quad (3.7)$$

By the first inequality of $\widetilde{(H+)}$, we have:

$$E_T \tilde{g}(X_T) + \int_t^T \int_s^T E_r \tilde{h}(r, \Theta_r) dr ds \geq \underline{\tilde{g}} e^{-\text{sgn}(\tilde{g})KT} + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds \geq 0.$$

Using (3.7), we get:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} \left[E_T g'(X_T) + \int_t^T E_s h_x(s) ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[E_T \tilde{g}(X_T) + \int_t^T \int_s^T E_r \tilde{h}(r, X_r, Y_r) dr ds \middle| \mathcal{F}_t \right] \\ &\geq \mathbb{E}^{\mathbb{Q}} \left[E_T \tilde{g}(X_T) + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds \middle| \mathcal{F}_t \right] \\ &\geq \mathbb{E}^{\mathbb{Q}} \left[\left(E_T \tilde{g}(X_T) + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds \right) \mathbf{1}_A(X_T) \middle| \mathcal{F}_t \right] \\ &\geq \left(\underline{\tilde{g}}^A e^{-\text{sgn}(\underline{\tilde{g}}^A)KT} + \underline{\tilde{h}}(t) \int_t^T e^{-\text{sgn}(\underline{\tilde{h}}(s))Ks} (T-s) ds \right) \mathbb{Q}(X_T \in A | \mathcal{F}_t) > 0. \end{aligned}$$

Then, for any $t \in [0, T]$, if $\widetilde{(H+)}$ holds, the first component Y_t of the solution of BSDE (2.1) has a probability distribution which is absolutely continuous with respect to the Lebesgue measure. The proof is similar if $\widetilde{(H-)}$ is satisfied. □

Example 3.7. We go back to Example 3.5 with $g \equiv \text{Id.}$ and $h(s, x, y, z) = (s-2)x$ which does not depend on z . On the one hand, we know from (3.4) that for all $t \in (0, 1]$, the law of Y_t has a density except when $t = 0$ or $t = 2 - \sqrt{3}$. On the other hand, our conditions in Theorem 3.6 read:

$$\bar{g} = \underline{\tilde{g}} = \tilde{g}(x) = t, \quad \tilde{h}(t, x, y) = \bar{\tilde{h}}(t) = \underline{\tilde{h}}(t) = -1, \quad K = 0,$$

from which $\widetilde{(H+)}$ becomes:

$$t - \int_t^1 (1-s) ds = t - (1-t) + \frac{1}{2} - \frac{t^2}{2} = -\frac{t^2}{2} + 2t - \frac{1}{2} > 0,$$

and $\widetilde{(H-)}$ becomes:

$$t - \int_t^1 (1-s)ds = t - (1-t) + \frac{1}{2} - \frac{t^2}{2} = -\frac{t^2}{2} + 2t - \frac{1}{2} < 0.$$

We hence conclude in view of Theorem 3.6 that the law of Y_t has a density for every $t \in (0, 1] \setminus \{2 - \sqrt{3}\}$. Note also that in this particular example, Condition $\widetilde{(H+)}$ and Condition $\widetilde{(H-)}$ are sufficient and necessary to obtain the existence of a density for Y . Finally, we emphasize once more that the counterpart of Condition $\widetilde{(H-)}$ in [3, Theorem 3.6] gives that whenever $2t - 1 < 0$, Y_t admits a density, which is clearly satisfied for $t = 2 - \sqrt{3}$. However we know that $Y_{2-\sqrt{3}} = 0$. This example proves that [3, Theorem 3.6] has to be modified.

3.2 The quadratic case

We now turn to the quadratic case and provide an extension of Theorem 3.3. Note however that the assumptions of this theorem do not find immediate counterparts in the quadratic setup since the latter involves the Lipschitz constant of h with respect to the z variable (see Remark 3.10). We also emphasize that unlike in the previous section where we merely extended existing results on the existence of densities for the Y component in the Lipschitz framework, the quadratic case we consider here was open until now.

Theorem 3.8. *Fix $t \in (0, T]$ and assume that $(A)(i)$, (Q) and $(D1)$ hold. If there is $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A \mid \mathcal{F}_t) > 0$ and one of the following assumptions holds (see Definitions (3.2)-(3.3))*

$(Q+)$ $g' \geq 0$ and $g'_A > 0$, $\mathcal{L}(X_T)$ -a.e. and $\underline{h}(t) \geq 0$,

$(Q-)$ $g' \leq 0$, $g'_A < 0$, $\mathcal{L}(X_T)$ -a.e. and $\bar{h}(t) \leq 0$,

then Y_t has a law absolutely continuous with respect to the Lebesgue measure.

Proof. To simplify the notations for any s in $[0, T]$, we set $\Theta_s := (X_s, Y_s, Z_s)$. We set $K := k_b \vee k_y \vee k_\sigma$. We assume that $(Q+)$ is satisfied (the proof with $(Q-)$ follows the same lines, so we omit it). According to Bouleau-Hirsch's criterion, it is enough to show that $\gamma_{Y_t} := \int_0^T |D_r Y_t|^2 dr > 0$, \mathbb{P} -a.s. As in the proof of Theorem 3.6,

$$\gamma_{Y_t} = \left(\mathbb{E} \left[g'(X_T) \zeta_T \psi_T + \int_t^T \psi_s h_x(s, \Theta_s) \zeta_s ds \mid \mathcal{F}_t \right] \right)^2 (\psi_t^{-1})^2 \int_0^t (\zeta_r^{-1} \sigma(r, X_r))^2 dr,$$

with

$$\psi_t \zeta_t = \underbrace{e^{\int_0^t (b_x(s, X_s) + h_y(s, \Theta_s) + \sigma_x(s, X_s) h_z(s, \Theta_s)) ds}}_{=: E_t} \underbrace{e^{\int_0^t (\sigma_x(s, X_s) + h_z(s, \Theta_s)) dW_s - \frac{1}{2} \int_0^t (\sigma_x(s, X_s) + h_z(s, \Theta_s))^2 ds}}_{=: M_t}.$$

Let \mathbb{Q} the probability measure equivalent to \mathbb{P} with density $\frac{d\mathbb{Q}}{d\mathbb{P}} := M_T$. Indeed, M is a martingale as $\int_0^\cdot (\sigma_x(s, X_s) + h_z(s, \Theta_s)) dW_s$ is a BMO martingale due to the boundedness of σ_x (by (A)) and the fact that $|h_z(s, \Theta_s)| \leq C(1 + |Z_s|)$ (by (Q)) and from the BMO property of $\int_0^\cdot Z_s dW_s$ (by Proposition 3.1). We therefore have:

$$\mathbb{E} \left[g'(X_T) \psi_T \zeta_T + \int_t^T \psi_s h_x(s, \Theta_s) \zeta_s ds \mid \mathcal{F}_t \right] = M_t \mathbb{E}^{\mathbb{Q}} \left[g'(X_T) E_T + \int_t^T h_x(s, \Theta_s) E_s ds \mid \mathcal{F}_t \right].$$

Using (Q+), we know that:

$$g'(X_T)E_T + \int_t^T h_x(s, \Theta_s)E_s ds \geq \underline{g}E_T + \underline{h}(t) \int_t^T E_s ds \geq 0.$$

Thus,

$$\begin{aligned} & \mathbb{E} \left[g'(X_T)\psi_T\zeta_T + \int_t^T \psi_s h_x(s, \Theta_s)\zeta_s ds \middle| \mathcal{F}_t \right] \\ & \geq M_t \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{X_T \in A} \left(g'(X_T)E_T + \int_t^T h_x(s, \Theta_s)E_s ds \right) \middle| \mathcal{F}_t \right] \\ & \geq M_t \left(\underline{g}^A e^{-2KT} \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{X_T \in A} e^{-K \int_0^T |h_z(s, \Theta_s)| ds} \middle| \mathcal{F}_t \right] \right. \\ & \quad \left. + \underline{h}(t) e^{-2KT} (T-t) \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{X_T \in A} e^{-K \int_0^T |h_z(s, \Theta_s)| ds} \middle| \mathcal{F}_t \right] \right) \\ & \geq M_t \left(\underline{g}^A e^{-2KT} \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{X_T \in A} e^{-K \sqrt{T} \sqrt{\int_0^T |h_z(s, \Theta_s)|^2 ds}} \middle| \mathcal{F}_t \right] \right. \\ & \quad \left. + \underline{h}(t) e^{-2KT} (T-t) \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{X_T \in A} e^{-K \sqrt{T} \sqrt{\int_0^T |h_z(s, \Theta_s)|^2 ds}} \middle| \mathcal{F}_t \right] \right), \end{aligned}$$

where the last inequality is due to Cauchy-Schwarz inequality. Besides, according to Assumption (Q), $|h_z(s, \Theta_s)| \leq C(1 + |Z_s|)$. Then, we deduce that $\int_0^T |h_z(s, \Theta_s)|^2 ds < +\infty$, \mathbb{P} -a.s. since $Z \in \mathbb{H}^2$. Hence, $M_t > 0$, \mathbb{P} -a.s. Given that the law of X_T is absolutely continuous with respect to the Lebesgue measure, we deduce that $\mathbb{E} \left[g'(X_T)\psi_T\zeta_T + \int_t^T \psi_s h_x(s, \Theta_s)\zeta_s ds \middle| \mathcal{F}_t \right] > 0$, \mathbb{P} -a.s. We conclude using Theorem 2.4. \square

Remark 3.9. Under (A)(i), (Q) or (L), and (D1) and if $g' \geq 0$ and $\underline{h}(t) \geq 0$ (resp. $g' \leq 0$ and $\bar{h}(t) \leq 0$) for $t \in [0, T]$, we deduce that for all $0 < r \leq t \leq T$, $D_r Y_t \geq 0$ (resp. $D_r Y_t \leq 0$) and the inequality is strict if there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and $g'_A > 0$ (resp. $g'_A < 0$).

Remark 3.10. Conditions (Q+) and (Q-) are stronger than (H+) and (H-), due to the unboundedness of h_z , which prevents us from reproducing the same proof than in [3]. Indeed, in this framework the quantity appearing for instance in (H+) becomes:

$$\underline{g} e^{-2K \operatorname{sgn}(\underline{g})T} e^{-K \operatorname{sgn}(\underline{g}) \int_0^T |h_z(s)| ds} + \underline{h}(t) e^{-2K \operatorname{sgn}(\underline{h}(t))T} \int_t^T e^{-K \operatorname{sgn}(\underline{h}(t)) \int_0^s |h_z(s)| ds},$$

whose sign for every $K \geq 0$ depends strongly on those of g' and h_x . This is why we must use the stronger conditions (Q+) and (Q-).

Remark 3.11. In [7, Corollary 3.5] comonotonicity conditions on the data of a BSDE under Assumption (Q) are given so that $Z_t \geq 0$, \mathbb{P} -a.s., $\forall t \in [0, T]$. In addition, the authors claim that strict comonotonicity entails that $Z_t > 0$, which implies by Bouleau-Hirsch criterion that the law of Y_t has a density with respect to the Lebesgue measure. However, we do not understand their proof and it is not true that an increasing mapping which is differentiable has a positive derivative everywhere (even if one relaxes it by asking for a positive derivative λ -almost everywhere) and one needs an extra assumption to prove that the derivative does not vanish. Indeed, take any closed set of positive Lebesgue measure with empty interior (for instance the Smith-Volterra-Cantor set on \mathbb{R}). By Whitney's extension Theorem there exists a differentiable increasing map whose derivative vanishes on this set.

4 Existence of a density for the control variable Z

We now turn to the problem of existence of a density for the marginal laws of Z . This question was studied in [1] when the generator is linear in z , that is to say $h(t, x, y, z) = \tilde{h}(t, x, y) + \alpha z$, which is from our point of view a too stringent assumption since by a Girsanov transformation this equation basically reduces to a BSDE with a generator which does not depend on z . In this section, we consider a general function h satisfying Assumption (L) or (Q). The following result will be crucial for us, and relies heavily on the Markovian framework we are working with.

Proposition 4.1 ([19, 12]). *Let Assumptions (A)(i), (L) or (Q) and (D1) hold, then there exists a map $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ in $\mathcal{C}^{1,2}$ such that*

$$Y_t = u(t, X_t), \quad t \in [0, T], \quad \mathbb{P} - a.s.$$

In addition, Z admits a continuous version given by

$$Z_t = u_x(t, X_t)\sigma(t, X_t), \quad t \in [0, T], \quad \mathbb{P} - a.s. \quad (4.1)$$

Remark 4.2. *Although the above proposition is completely proved in [19] in the Lipschitz case, we did not find a proper reference in the quadratic case, except for [12] which proves the result under Assumption (Q), with the exception that u is only shown to be in $\mathcal{C}^{1,1}$. Nonetheless, one can still obtain the required result by proving that Theorem 3.1 of [19] still holds for a BSDE with a driver which is uniformly Lipschitz in y and stochastic Lipschitz in z with a Lipschitz process in $\mathbb{H}_{\text{BMO}}^2$ (which is exactly the case of the BSDE satisfied by the Malliavin derivative of Y). This can be achieved by following exactly the steps of the proof of Theorem 3.1 in [19], where the a priori estimates of their Lemma 2.2 have to be replaced by those given in Lemma A.1 of [12].*

Note that by definition, Z is an element of \mathbb{H}^2 . As a consequence, for any fixed element t in $[0, T]$, the random variable Z_t is not uniquely defined, which makes the density analysis ill-posed. However, by the previous proposition, Z admits in our framework a continuous version. From now on, we will always consider this version. In view of Proposition 4.1, the chain rule formula implies that Y_t belongs to $\mathbb{D}^{2,2}$ and

$$D^2 Y_t = u_x(t, X_t)D^2 X_t + u_{xx}(t, X_t)(DX_t)^{\otimes 2}, \quad \mathbb{P} - a.s. \quad (4.2)$$

The following Lemma is due to Ma and Zhang in [19, Lemma 2.4] in the Lipschitz case (and to Pardoux and Peng [24] for the representation of Z as a Malliavin trace of Y see (4.3) below), and can readily be extended to the quadratic case using the same arguments as in Remark 4.2 above.

Lemma 4.3. *Under assumptions (A)(i), (L) or (Q), (D1) and (D2), there exists a version of $(D_r X_t, D_r Y_t, D_r Z_t)$ for all $0 < r \leq t \leq T$ which satisfies:*

$$\begin{aligned} D_r X_t &= \nabla X_t (\nabla X_r)^{-1} \sigma(r, X_r), \quad D_r Y_t = \nabla Y_t (\nabla X_r)^{-1} \sigma(r, X_r), \quad D_r Z_t = \nabla Z_t (\nabla X_r)^{-1} \sigma(r, X_r), \\ Z_t &= D_t Y_t := \lim_{s \nearrow t} D_s Y_t, \quad \mathbb{P} - a.s., \quad \text{for a.e. } t \in [0, T], \end{aligned} \quad (4.3)$$

where $(\nabla X, \nabla Y, \nabla Z)$ is the solution to the following FBSDE:

$$\begin{cases} \nabla X_t = \int_0^t b_x(s, X_s) \nabla X_s ds + \int_0^t \sigma_x(s, X_s) \nabla X_s dW_s, \\ \nabla Y_t = g'(X_T) \nabla X_T + \int_t^T \nabla h(s, \Theta_s) \cdot \nabla \Theta_s ds - \int_t^T \nabla Z_s dW_s. \end{cases} \quad (4.4)$$

4.1 The Lipschitz case

Let $t \in (0, T]$ and $A \in \mathcal{B}(\mathbb{R})$. We set:

$$\underline{g}'' := \min_{x \in \mathfrak{S}(X_T)} g''(x), \quad \underline{g}''^A := \min_{x \in \mathfrak{S}(X_T) \cap A} g''(x), \quad \underline{g}' := \min_{x \in \mathfrak{S}(X_T)} g'(x), \quad \underline{g}'^A := \min_{x \in \mathfrak{S}(X_T) \cap A} g'(x),$$

$$\underline{h}_{xx}(t) := \min_{t \in [t, T], (x, y, z) \in \mathbb{R}^3} h_{xx}(t, x, y, z).$$

Theorem 4.4. *Let Assumption (A)(i), (L) and (D2) hold. Let $0 < t \leq T$ and assume moreover*

- *There exist $(\underline{a}, \bar{a}) \in (0, +\infty)$, such that $\underline{a} \leq D_r X_u \leq \bar{a}$, for all $0 < r < u \leq T$,*
- *There exists $\bar{b} \geq 0$, such that $0 \leq D_{r,t}^2 X_u \leq \bar{b}$, for all $0 < r, t < u \leq T$,*
- *(C+) holds*
- *$h_{xy} = 0$ or ($h_{xy} \geq 0$ and $g' \geq 0$, $\mathcal{L}(X_T) - a.e.$).*

If there exists a set $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and such that

$$\mathbf{1}_{\{\underline{g}'' < 0\}} \underline{g}'' \bar{a}^2 + \underline{g}' \mathbf{1}_{\{\underline{g}' < 0\}} \bar{b} + (\mathbf{1}_{\{\underline{g}'' \geq 0\}} \underline{g}'' + \underline{h}_{xx}(t)(T-t)) \underline{a}^2 \geq 0,$$

and

$$(\mathbf{1}_{\{\underline{g}''^A < 0\}} \underline{g}''^A \bar{a}^2 + \underline{g}'^A \mathbf{1}_{\{\underline{g}' < 0\}} \bar{b}) + (\mathbf{1}_{\{\underline{g}''^A \geq 0\}} \underline{g}''^A + \underline{h}_{xx}(t)(T-t)) \underline{a}^2 > 0,$$

then, the law of Z_t has a density with respect to the Lebesgue measure.

Proof. Under the assumptions of Theorem 4.4, we obtain for $0 < r, s < t \leq T$:

$$\begin{aligned} D_{r,s}^2 Y_t &= g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T - \int_t^T D_{r,s}^2 Z_u dW_u \\ &+ \int_t^T [h_x(u, \Theta_u) D_{r,s}^2 X_u + h_{xx}(u, \Theta_u) D_r X_u D_s X_u + h_{xy}(u, \Theta_u) D_s X_u D_r Y_u \\ &+ h_y(u, \Theta_u) D_{r,s}^2 Y_u + h_{xy}(u, \Theta_u) D_r X_u D_s Y_u + h_{yy}(u, \Theta_u) D_r Y_u D_s Y_u \\ &+ h_z(u, \Theta_u) D_{r,s}^2 Z_u + h_{zz}(u, \Theta_u) D_r Z_u D_s Z_u] du. \end{aligned}$$

Let $\tilde{\mathbb{P}}$ be the probability equivalent to \mathbb{P} such that¹:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left(\int_0^T h_z(s, \Theta_s) dW_s - \frac{1}{2} \int_0^T |h_z(s, \Theta_s)|^2 ds \right), \quad (4.5)$$

where h_z is bounded thanks to Assumption (L). Under $\tilde{\mathbb{P}}$ defined by (4.5), we obtain:

$$\begin{aligned} D_{r,s}^2 Y_t &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T \right. \\ &+ \int_t^T [h_x(u, \Theta_u) D_{r,s}^2 X_u + h_{xx}(u, \Theta_u) D_r X_u D_s X_u + h_{xy}(u, \Theta_u) D_s X_u D_r Y_u \\ &+ h_y(u, \Theta_u) D_{r,s}^2 Y_u + h_{xy}(u, \Theta_u) D_r X_u D_s Y_u + h_{yy}(u, \Theta_u) D_r Y_u D_s Y_u \\ &+ h_{zz}(u, \Theta_u) D_r Z_u D_s Z_u] du \left. \middle| \mathcal{F}_t \right]. \end{aligned}$$

¹It is exactly here that the proof change under Assumption (Q). Indeed, under these assumption $\int_0^t h_z(s, \Theta_s) dW_s$ is a BMO-martingale, then we can define $\tilde{\mathbb{P}}$ as in (4.5).

By standard linearization techniques, we obtain:

$$\begin{aligned} D_{r,s}^2 Y_t &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{\int_t^T h_y(u, \Theta_u) du} (g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T) \right. \\ &\quad + \int_t^T e^{\int_t^u h_y(v, \Theta_v) dv} [h_x(u, \Theta_u) D_{r,s}^2 X_u + h_{xx}(u, \Theta_u) D_r X_u D_s X_u \\ &\quad + h_{xy}(u, \Theta_u) (D_r X_u D_s Y_u + D_s X_u D_r Y_u) \\ &\quad \left. + h_{yy}(u, \Theta_u) D_r Y_u D_s Y_u + h_{zz}(u, \Theta_u) D_r Z_u D_s Z_u] du \middle| \mathcal{F}_t \right]. \end{aligned}$$

Then, using Remark 3.9, Lemma 4.3 and our assumptions we obtain:

$$\begin{aligned} &e^{\int_t^T h_y(u, \Theta_u) du} (g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T) \\ &+ \int_t^T e^{\int_t^u h_y(v, \Theta_v) dv} [h_x(u, \Theta_u) D_{r,s}^2 X_u + h_{xx}(u, \Theta_u) D_r X_u D_s X_u \\ &+ h_{xy}(u, \Theta_u) (D_r X_u D_s Y_u + D_s X_u D_r Y_u) \\ &+ h_{yy}(u, \Theta_u) D_r Y_u D_s Y_u + h_{zz}(u, \Theta_u) D_r Z_u D_s Z_u] du \\ &\geq e^{\int_t^T h_y(u, \Theta_u) du} \left(\mathbf{1}_{\{g'' < 0\}} \underline{g''} \bar{a}^2 + \underline{g'} \mathbf{1}_{\{g' < 0\}} \bar{b} + (\mathbf{1}_{\{g'' \geq 0\}} \underline{g''} + \underline{h_{xx}}(t)(T-t)) \underline{a}^2 \right) \geq 0. \end{aligned}$$

We deduce that:

$$\begin{aligned} D_{r,s}^2 Y_t &\geq \mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{\int_t^T h_y(u, \Theta_u) du} \mathbf{1}_{X_T \in A} (g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T) \right. \\ &\quad \left. + \mathbf{1}_{X_T \in A} \int_t^T e^{-K(u-t)} [h_{xx}(u, \Theta_u) \underline{a}^2] du \middle| \mathcal{F}_t \right] \\ &\geq e^{-KT} \left(\mathbf{1}_{\{\underline{g''}^A < 0\}} \underline{g''}^A \bar{a}^2 + \underline{g'}^A \mathbf{1}_{\{\underline{g'} < 0\}} \bar{b} \right) \tilde{\mathbb{P}}(X_T \in A | \mathcal{F}_t) \\ &\quad + e^{-KT} \left(\mathbf{1}_{\{\underline{g''}^A \geq 0\}} \underline{g''}^A + \underline{h_{xx}}(t)(T-t) \right) \underline{a}^2 \tilde{\mathbb{P}}(X_T \in A | \mathcal{F}_t). \end{aligned}$$

Using the fact that $D^2 Y_t$ is symmetric, the chain rule formula, (4.1) and (4.2) and the fact that $\lim_{s \nearrow t} D_{r,s}^2 X_t = \sigma'(t, X_t) D_r X_t$, we have that $\lim_{s \nearrow t} D_{r,s}^2 Y_t = D_r Z_t$, from which we deduce that $D_r Z_t > 0$, \mathbb{P} -a.s. Then according to Bouleau and Hirsch's Theorem, we conclude that the law of Z_t has a density with respect to the Lebesgue measure. \square

Remark 4.5. One can provide an alternative version of the previous result, whose proof follows the same lines as the one of Theorem 4.4. Fix t in $(0, T]$. Let Assumptions (L), (A)(i) and (D2) hold and assume that there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$, and such that one of the two following conditions is satisfied:

- a) (A)(ii+) and (C+) hold true and $g'' \geq 0$, $g''|_A > 0$ and $g' \geq 0$, $\mathcal{L}(X_T)$ -a.e.
- b) (A)(ii-) and (C-) hold true and $g'' \leq 0$, $g''|_A < 0$ and $g' \leq 0$, $\mathcal{L}(X_T)$ -a.e.,

then, for all $t \in (0, T]$, the law of Z_t has a density with respect to Lebesgue measure.

Remark 4.6. Note that according to the first step of the proof of Theorem 4.3. in [1], Condition (A)(ii+) (resp. (A)(ii-)) ensures that $D^2 X$ is non-negative (resp. non-positive).

When Assumption (M) holds, Theorem 4.4 takes a different form as shown below in Theorem 4.8, mainly because of the following easy result.

Proposition 4.7. *Under Assumptions (M), (L) or (Q) and (D2), for all $0 < r, s \leq t \leq T$ we have $D_r Y_t = D_s Y_t = Z_t$ and $D_r Z_t = D_s Z_t$, \mathbb{P} -a.s.*

Proof. Once again we set $\Theta_s := (X_s, Y_s, Z_s)$. We know that for all $0 < r \leq t \leq T$:

$$\begin{aligned} D_r Y_t &= g'(X_T) f'(T, W_T) + \int_t^T (h_x(s, \Theta_s) f'(s, W_s) + h_y(s, \Theta_s) D_r Y_s + h_z(s, \Theta_s) D_r Z_s) ds \\ &\quad - \int_t^T D_r Z_s dW_s. \end{aligned}$$

Then $(D_r Y, D_r Z)$ satisfies a linear BSDE which does not depend on r and by the unicity of the solution we deduce that for all $0 < r, s \leq t \leq T$ we have $D_r Y_t = D_s Y_t$ and $D_r Z_t = D_s Z_t$, \mathbb{P} -a.s. Finally, $D_r Y_t = Z_t$ by (4.3). □

Let us now consider the following assumptions:

($\tilde{C}+$) $h_{zz} \geq 0$ and $h_{xz} = h_{yz} \equiv 0$.

($\tilde{C}-$) $h_{zz} \geq 0$ and $h_{xz} = h_{yz} \equiv 0$.

Under Assumption ($\tilde{C}+$) or ($\tilde{C}-$), we recall that:

$$\begin{aligned} D_r Z_t &= g''(X_T) |f'(T, W_T)|^2 + g'(X_t) f''(T, W_T) \\ &\quad + \int_t^T \left[h_x(u, \Theta_u) f''(u, W_u) + h_y(u, \Theta_u) D_{r,t}^2 Y_u \right] du \\ &\quad + \int_t^T \left[|f'(u, W_u)|^2 h_{xx}(u, \Theta_u) + (h_{xy}(u, \Theta_u) D_r Y_u + D_t Y_u h_{xy}(u, \Theta_u)) f'(u, W_u) \right] du \\ &\quad + \int_t^T \left[h_{yy}(u, \Theta_u) \underbrace{D_t Y_u D_r Y_u}_{=|Z_u|^2} + \underbrace{D_t Z_u D_r Z_u}_{=|D_r Z_u|^2} h_{zz}(u, \Theta_u) \right] du - \int_t^T D_{r,t}^2 Z_u d\tilde{W}_u, \end{aligned}$$

with $\tilde{W} := W - \int_0^\cdot h_z(s, \Theta_s) ds$. We set $\theta = (x, y, z)$, and

$$\begin{aligned} \tilde{h}(t, w, x, y, z, \tilde{z}) &:= h_{xx}(t, \theta) |f'(t, w)|^2 + h_x(t, \theta) f''(t, w) + (h_{yy}(t, \theta) z + 2h_{xy}(t, \theta) f'(t, w)) z \\ &\quad + h_y(t, \theta) \tilde{z}, \end{aligned}$$

$$\underline{\tilde{h}}(t) = \min_{(s, w, x, y, z, \tilde{z}) \in [t, T] \times \mathbb{R}^5} \tilde{h}(s, w, x, y, z, \tilde{z}), \quad \overline{\tilde{h}}(t) = \max_{(s, w, x, y, z, \tilde{z}) \in [t, T] \times \mathbb{R}^5} \tilde{h}(s, w, x, y, z, \tilde{z}).$$

Theorem 4.8. *Assume that (M), (L) and (D2) are satisfied and that there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and one of the two following assumptions holds:*

- a) *Assumption ($\tilde{C}+$), $((g' \circ f) f')' + (T - t) \underline{\tilde{h}}(t) \geq 0$ and $((g' \circ f) f')'_A + (T - t) \underline{\tilde{h}}(t) > 0$.*
- b) *Assumption ($\tilde{C}-$), $\overline{((g' \circ f) f')'} + (T - t) \overline{\tilde{h}}(t) \leq 0$ and $\overline{((g' \circ f) f')'_A} + (T - t) \overline{\tilde{h}}(t) < 0$.*

Then, the law of Z_t is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

Proof. Using Proposition 4.7, we recall that:

$$\begin{aligned} D_r Z_t &= g''(X_T) |f'(T, W_T)|^2 + g'(X_t) f''(T, W_T) \\ &\quad + \int_t^T \tilde{h}(u, W_u, X_u, Y_u, Z_u, D_r Z_u) + |D_r Z_u|^2 h_{zz}(u) du - \int_t^T D_{r,t}^2 Z_u d\tilde{W}_u, \end{aligned}$$

where $\tilde{W} := W - \int_0^\cdot h_z(u, \Theta_u) du$. Then the proof follows exactly the same line as the one of Theorem 4.4. \square

Remark 4.9. Condition (M) is not incompatible with the fact that the process X is a solution to a SDE. Indeed, let σ be in $\mathcal{C}^2(\mathbb{R})$ with bounded first and second derivatives. Then, according to Doss and Sussmann's Theorem (see [8], [13, Proposition 5.2.21]), the one-dimensional SDE

$$X_t = X_0 + \int_0^t \frac{1}{2} \sigma(X_s) \sigma'(X_s) ds + \int_0^t \sigma(X_s) dW_s,$$

has a unique solution, given by $X_t = \varphi(t, W_t)$, $0 \leq t < \infty$, where $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function satisfying $\frac{\partial \varphi}{\partial x} = \sigma(\varphi)$, $\varphi(0, x) = x$.

4.2 The quadratic case

In this section, we obtain existence results for the density of Z under Assumption (Q). We actually have exactly the same type of results as in the Lipschitz case with similar proofs, which highlights the robustness and flexibility of our approach. Let us detail first the changes that we have to make. Under (Q), using the fact that for all $s \in [0, T]$ $|h_z(s, \Theta_s)| \leq C(1 + |Z_s|)$ and according to Proposition 3.1 we deduce that $\int_0^\cdot h_z(s, \Theta_s) dW_s$ is a BMO-martingale. Then, according to Theorem 2.3 in [14], the stochastic exponential of $\int_0^\cdot h_z(s, \Theta_s) dW_s$ is a uniformly integrable martingale and we can apply Girsanov's Theorem. We also emphasize that in (Q), g is not assumed to be twice continuously differentiable. Indeed, to recover the BMO properties linked to quadratic BSDEs (and thus in order to be able to apply the above reasoning), g needs to be bounded, which is incompatible with g convex (or concave). Nevertheless, there exist terminal conditions g which are twice differentiable almost everywhere on the support of the law of X_T (which is some closed subset of \mathbb{R}), such that their second-order derivative have a given sign there.

Theorem 4.10. Let Assumption (A), (Q) and (D2) hold. Let $0 < t \leq T$ and assume moreover

- There exist (\underline{a}, \bar{a}) s.t., $0 < \underline{a} \leq D_r X_u \leq \bar{a}$, for all $0 < r < u \leq T$.
- There exists \bar{b} s.t., $0 \leq D_{r,s}^2 X_u \leq \bar{b}$, for all $0 < r, s < u \leq T$.
- (C+) holds and $h_y \geq 0$.
- $h_{xy} = 0$ or $(h_{xy} \geq 0 \text{ and } g' \geq 0, \mathcal{L}(X_T)\text{-a.e.})$.

If there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t)$ and such that:

$$\mathbf{1}_{\{g'' < 0\}} g'' \underline{a}^2 + g' \mathbf{1}_{\{g' < 0\}} \bar{b} + (\mathbf{1}_{\{g'' \geq 0\}} g'' + h_{xx}(t)(T-t)) \underline{a}^2 \geq 0,$$

and

$$(\mathbf{1}_{\{g''^A < 0\}} g''^A \underline{a}^2 + g'^A \mathbf{1}_{\{g' < 0\}} \bar{b}) + (\mathbf{1}_{\{g''^A \geq 0\}} g''^A + h_{xx}(t)(T-t)) \underline{a}^2 > 0,$$

then, the law of Z_t has a density with respect to the Lebesgue measure.

Proof. As in the proof of Theorem 4.4, we notice that for all $0 < r, t \leq s \leq T$:

$$\begin{aligned} D_{r,s}^2 Y_t &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[g''(X_T) D_r X_T D_s X_T + g'(X_T) D_{r,s}^2 X_T \right. \\ &\quad + \int_t^T [h_x(u, \Theta_u) D_{r,s}^2 X_u + h_{xx}(u, \Theta_u) D_r X_u D_s X_u + h_y(u, \Theta_u) D_r Y_u D_s Y_u \\ &\quad \left. + h_{yy}(u, \Theta_u) D_{r,s}^2 Y_u + h_{zz}(u, \Theta_u) D_r Z_u D_s Z_u] du \middle| \mathcal{F}_t \right], \end{aligned}$$

where $\tilde{\mathbb{P}}$ is the equivalent probability measure to \mathbb{P} with density

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp \left(\int_0^T h_z(u, \Theta_u) dW_u - \frac{1}{2} \int_0^T |h_z(u, \Theta_u)|^2 du \right),$$

given that $\int_0^\cdot h_z(u, \Theta_u) dW_u$ is a BMO-martingale and using Theorem 2.3 in [14]. Then the proof is similar to that of Theorem 4.4. \square

We give also a theorem under Assumption (M):

Theorem 4.11. *Assume that (M), (Q) and (D2) are satisfied and that there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mathbb{P}(X_T \in A | \mathcal{F}_t) > 0$ and one of the two following assumptions holds:*

- a) *Assumption $(\tilde{C}+)$, $\underline{((g' \circ f)f')'} + (T-t)\tilde{h}(t) \geq 0$ and $\underline{((g' \circ f)f')'_A} + (T-t)\tilde{h}(t) > 0$.*
- b) *Assumption $(\tilde{C}-)$, $\overline{((g' \circ f)f')'} + (T-t)\bar{h}(t) \leq 0$ and $\overline{((g' \circ f)f')'_A} + (T-t)\bar{h}(t) < 0$.*

Then, the law of Z_t is absolutely continuous with respect to the Lebesgue measure.

The proof is the same as the proof of Theorem 4.8 using the BMO property of $\int_0^\cdot Z_s dW_s$, we therefore omit it. We now turn to the simplest case of quadratic growth BSDE and verify that it is covered by our result.

Example 4.12. *Let us consider the following BSDE*

$$Y_t = g(W_T) + \int_t^T \frac{1}{2} |Z_s|^2 ds - \int_t^T Z_s dW_s,$$

where g is bounded. According to Theorem 4.10 with $\bar{a} = \underline{a} = 1$, $\bar{b} = 0$ and $h_{xx} = 0$, we deduce that for all $t \in (0, T]$, the law of Z_t has a density with respect to the Lebesgue measure if $g'' \geq 0$, $\lambda(dx)$ -a.e. and if there exists $A \in \mathcal{B}(\mathbb{R})$ with positive Lebesgue measure such that $g''_A > 0$.

We emphasize that, as a sanity check, this can be verified by direct calculations. Indeed, using the fact that if $F \in \mathbb{D}^{1,2}$ then $D_r(\mathbb{E}[F | \mathcal{F}_t]) = \mathbb{E}[D_r F | \mathcal{F}_t] \mathbf{1}_{[0,t)}(r)$ (see [21, Proposition 1.2.4]) we deduce that if $0 \leq r < t \leq T$ then:

$$D_r Y_t = \frac{\mathbb{E}[g'(W_T) e^{g(W_T)} | \mathcal{F}_t]}{\mathbb{E}[e^{g(W_T)} | \mathcal{F}_t]},$$

which does not depend on r . Then according to Proposition 4.3, $Z_t = \frac{\mathbb{E}[g'(W_T) e^{g(W_T)} | \mathcal{F}_t]}{\mathbb{E}[e^{g(W_T)} | \mathcal{F}_t]}$.

Take $0 < r < t \leq T$, then:

$$D_r Z_t = \frac{\mathbb{E}[g''(W_T) e^{g(W_T)} + |g'(W_T)|^2 e^{g(W_T)} | \mathcal{F}_t] \mathbb{E}[e^{g(W_T)} | \mathcal{F}_t] - |\mathbb{E}[g'(W_T) e^{g(W_T)} | \mathcal{F}_t]|^2}{\mathbb{E}[e^{g(W_T)} | \mathcal{F}_t]}.$$

Using Cauchy-Schwarz inequality, if $g'' \geq 0$, $\lambda(dx)$ -a.e. and if there exists $A \in \mathcal{B}(\mathbb{R})$ with positive Lebesgue measure such that $g''_A > 0$, we deduce that for all $t \in (0, T]$, Z_t has a density with respect to the Lebesgue measure by Theorem 2.4.

5 Density estimates for the marginal laws of Y and Z

Up to now, the density estimates obtained in the literature relied mainly on the fact that the framework considered implied that the Malliavin derivative of Y was bounded. Hence, using the Nourdin-Viens' formula (or more precisely their Corollary 3.5 in [20]), it could be showed that the law of Y has Gaussian tails. Although such an approach is perfectly legitimate from the theoretical point of view, let us start by explaining why, as pointed out in the introduction, we think that this is not the natural framework to work with when dealing with BSDEs. Consider indeed the following example.

Example 5.1. *Let us consider the FBSDE (2.1), with $T = 1$, $g(x) := x^3$, $h(t, x, y, z) := 3x$, $b(t, x) = 0$, $\sigma(t, x) = 1$ and $X_0 = 0$. Then, simple computations show that the unique solution is given by*

$$X_t = W_t, \quad Y_t = W_t^3 + 6W_t(1 - t), \quad Z_t = 3W_t^2 + 6(1 - t).$$

Then, both Y_t and Z_t have a law which is absolutely continuous with respect to the Lebesgue measure, for every $t \in (0, 1]$, but neither Y_t nor Z_t has Gaussian tails.

Moreover, when it comes to applications dealing with generators with quadratic growth, assuming that the Malliavin derivative of Y is bounded implies that the process Z itself is bounded as $Z_t = D_t Y_t$, which is seldom satisfied in applications, since in general, one only knows that $Z \in \mathbb{H}_{\text{BMO}}^2$.

One of the main applications of the results we obtain in this section is the precise analysis of the error in the truncation method in numerical schemes for quadratic BSDEs, introduced in [11] and studied in [5]. We recall that according to Proposition 4.1 there exists a function $v : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ in $\mathcal{C}^{1,2}$ such that $Y_t = v(t, X_t)$ and $Z_t = v_x(t, X_t)$. Since we want to study the tails of the laws of Y and Z , we will assume from now on that the support of these law is \mathbb{R} , which implies that neither v nor v' is bounded from below or above. Moreover, we emphasize that throughout this section, we will assume that Y_t and Z_t do have a law which is absolutely continuous, so as to highlight the conditions needed to obtain the estimates. Throughout this section we assume that $X_t = W_t$ in (2.1) (that is $X_0 = 0$, $\sigma \equiv 1$, $b \equiv 0$).

5.1 Preliminary results

We will have to study the asymptotic growth of v and v_x in the neighborhood of $\pm\infty$. To this end, we introduce for any measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following two kinds of growth rates:

$$\overline{\alpha}_f := \inf \left\{ \alpha > 0, \limsup_{|x| \rightarrow +\infty} \left| \frac{f(x)}{x^\alpha} \right| < +\infty \right\}, \quad \underline{\alpha}_f := \inf \left\{ \alpha > 0, \liminf_{|x| \rightarrow +\infty} \left| \frac{f(x)}{x^\alpha} \right| < +\infty \right\}.$$

Lemma 5.2. *Let $f \in \mathcal{C}^1(\mathbb{R})$. Assume that for all $x \in \mathbb{R}$, $f'(x) > 0$. If $0 < \underline{\alpha}_f < +\infty$ then for all positive constant $0 < \eta < \underline{\alpha}_f$:*

$$\overline{\alpha}_{f^{(-1)}} \leq \frac{1}{\underline{\alpha}_f - \eta},$$

where $f^{(-1)}$ is the inverse function of f .

Proof. Using the definition of $\underline{\alpha}_f$, we deduce that for all $\eta > 0$,

$$\liminf_{|x| \rightarrow +\infty} \left| \frac{f(x)}{x^{\underline{\alpha}_f - \eta}} \right| = \lim_{|x| \rightarrow +\infty} \left| \frac{f(x)}{x^{\underline{\alpha}_f - \eta}} \right| = +\infty.$$

Since f and $f^{(-1)}$ are increasing and unbounded from above and below, we deduce that there exists $\bar{x} > 0$ such that for all $x \geq \bar{x}$, $f(x)$ and $f^{(-1)}$ are positive. Then, for all $M > 0$, there exists $x_0 \geq \bar{x}$ such that for all $x \geq x_0 > 0$ and for all $y \geq Mx_0^{\frac{\alpha_f - \eta}{\alpha_f - \eta}} \vee \bar{x}$

$$f(x) \geq Mx^{\frac{\alpha_f - \eta}{\alpha_f - \eta}} \iff f\left((yM^{-1})^{\frac{1}{\alpha_f - \eta}}\right) \geq y \iff (yM^{-1})^{\frac{1}{\alpha_f - \eta}} \geq f^{(-1)}(y).$$

This implies directly that $\limsup_{y \rightarrow +\infty} \left| \frac{f^{(-1)}(y)}{y^{\frac{1}{\alpha_f - \eta}}} \right| < +\infty$. The proof is similar when y goes to $-\infty$. \square

It is rather natural to expect that for well-behaved functions $f \in \mathcal{C}^1(\mathbb{R})$, $\overline{\alpha_f} = \underline{\alpha_f}$ and $\overline{\alpha_{f'}} = \underline{\alpha_{f'}} + 1$. However, the situation is unfortunately not that clear. First of all, this may not be true if f is not monotone. Indeed, let $f(x) := x^2 \sin(x)$, then $\overline{\alpha_f} = \underline{\alpha_f} = 2$. Furthermore, the strict monotonicity of f is not sufficient either. Without being completely rigorous, let us describe a counterexample. Consider a function f defined on \mathbb{R}_+ , equal to the identity on $[0, 1]$, which then increases as x^4 until it crosses $x \mapsto x^2$ for the first time, which then increases as $x^{1/2}$ until it crosses $x \mapsto x$ for the first time and so on. Finally, extend it by symmetry to \mathbb{R}_- . Then, it can be checked that $\overline{\alpha_f} = 2$, $\underline{\alpha_f} = 1$, $\overline{\alpha_{f'}} = 3$, $\underline{\alpha_{f'}} = 0$.

A nice sufficient condition for the aforementioned result to hold is that f' is a *regularly varying functions* (see [4] and [25]).

Lemma 5.3. *Assume that f' is equivalent in $+\infty$ (resp. in $-\infty$) to a regularly varying function with Karamata's decomposition $x^\beta L_1(x)$ where L_1 is slowly varying (resp. $x^\beta L_2(x)$ where L_2 is slowly varying) and where $\beta > 0$. Then*

- (i) *f is equivalent in $+\infty$ (resp. in $-\infty$) to a regularly varying function with Karamata's decomposition $x^{\beta+1} \widetilde{L}_1(x)$ where \widetilde{L}_1 is slowly varying (resp. $x^{\beta+1} \widetilde{L}_2(x)$ where \widetilde{L}_2 is slowly varying).*
- (ii) $\overline{\alpha_f} = \underline{\alpha_f} = \underline{\alpha_{f'}} + 1 = \overline{\alpha_{f'}} + 1$.

Proof. By Karamata's Theorem (see Theorem 1.5.11 in [4] with $\sigma = 1$), for any $x_0 \in \mathbb{R}$:

$$\frac{xf'(x)}{f(x) - f(x_0)} \rightarrow \beta + 1, \quad \text{when } x \rightarrow +\infty. \quad (5.1)$$

In addition, f' is equivalent to a regularly varying function with Karamata's decomposition $x^\beta L_1(x)$ when $x \rightarrow +\infty$, hence in view of (5.1), there exists a function \widetilde{L}_1 (equivalent to a constant times L_1 at $+\infty$) slowly varying such that f is equivalent when $x \rightarrow +\infty$ to a regularly varying function with Karamata's decomposition $x^{\beta+1} \widetilde{L}_1(x)$. The same result holds when $x \rightarrow -\infty$. We now show (ii). According to Proposition 1.3.6 (v) in [4] and (i), we deduce that:

$$\overline{\alpha_f} = \beta + 1 = \underline{\alpha_f} \quad \text{and} \quad \overline{\alpha_{f'}} = \beta = \underline{\alpha_{f'}}.$$

\square

5.2 A general estimate

From now on, for a map $(t, x) \mapsto v(t, x)$, $v'(t, x)$ will denote for simplicity the derivative of v with respect to the space variable. The following general theorem gives us density estimates for the tails of the law of random variables of the form $v(t, W_t)$ and will be used to obtain estimates for the laws of Y_t and Z_t .

Theorem 5.4. Fix $t \in (0, T]$. Let $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ in $\mathcal{C}^{1,1}$ and let $P_t := v(t, W_t)$. Assume furthermore that $P_t \in L^1(\mathbb{P})$, that v is unbounded in x both from above and from below, that $v' > 0$, $\underline{\alpha}_v \in (0, +\infty)$, $\overline{\alpha}_{v'} < +\infty$ and that there exist $\tilde{\alpha} > 0$ and $K > 0$ such that:

$$\frac{1}{v'(t, x)} \leq K(1 + |x|^{\tilde{\alpha}}), \text{ for all } x \in \mathbb{R}. \quad (5.2)$$

Then, the law of P_t has a density with respect to the Lebesgue measure, denoted by ρ_t , and for all $\varepsilon, \varepsilon' > 0$ there exist two positive constants $M(\varepsilon')$ and $M'(\varepsilon, \varepsilon')$ such that for every $y \in \mathbb{R}$

$$\rho_t(y) \leq \frac{\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{2M(\varepsilon')t} \left(1 + |y|^{2\tilde{\alpha}(\overline{\alpha}_{v'}(-1) + \varepsilon')}\right) \exp \left(- \int_0^{y - \mathbb{E}[P_t]} \frac{(M'(\varepsilon, \varepsilon')t)^{-1} x dx}{1 + |x + \mathbb{E}[P_t]|^{2(\overline{\alpha}_{v'} + \varepsilon)(\overline{\alpha}_{v'}(-1) + \varepsilon')}} \right), \quad (5.3)$$

and

$$\rho_t(y) \geq \frac{(2M'(\varepsilon, \varepsilon')t)^{-1} \mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{1 + |y|^{2(\overline{\alpha}_{v'} + \varepsilon)(\overline{\alpha}_{v'}(-1) + \varepsilon')}} \exp \left(- \int_0^{y - \mathbb{E}[P_t]} \frac{x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha}_{v'}(-1) + \varepsilon')}\right) dx}{M(\varepsilon')t} \right). \quad (5.4)$$

Proof. Notice immediately that since the map $x \mapsto v(t, x)$ is in $\mathcal{C}^1(\mathbb{R})$ and increasing, the law of P_t clearly has a density. We prove inequalities (5.3) and (5.4) using Nourdin and Viens' formula (see Theorem 2.5). The rest of the proof is divided into three steps.

Step 1: Given that for all $0 < r \leq t \leq T$, $D_r P_t = v'(t, W_t)$, the function g_{P_t} defined by (2.5) becomes

$$g_{P_t}(y) := \int_0^\infty e^{-a} \mathbb{E} \left[\mathbb{E}^* [\langle \Phi_{P_t}(W), \widetilde{\Phi}_{P_t}^a(W) \rangle_{\mathfrak{H}}] | P_t - \mathbb{E}[P_t] = y \right] da, \quad y \in \mathbb{R},$$

with² $\Phi_{P_t}(W) := v'(t, W_t)$ and where $\widetilde{\Phi}_{P_t}^a(W) := \Phi_{P_t}(e^{-a}W + \sqrt{1 - e^{-2a}}W^*)$ with W^* an independent copy of W defined on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ where \mathbb{E}^* is the expectation under \mathbb{P}^* (Φ_{P_t} being extended on $\Omega \times \Omega^*$). Letting $\phi(z) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$, we get that

$$\begin{aligned} g_{P_t}(y) &= \int_0^\infty e^{-a} \mathbb{E} \left[\mathbb{E}^* [\langle \Phi_{P_t}(W), \widetilde{\Phi}_{P_t}^a(W) \rangle_{\mathfrak{H}}] | W_t = v^{(-1)}(t, y + \mathbb{E}[P_t]) \right] da, \quad y \in \mathbb{R}, \\ &= t v'(t, v^{(-1)}(t, y + \mathbb{E}[P_t])) \int_0^\infty e^{-a} \int_{\mathbb{R}} v' \left(t, e^{-a} v^{(-1)}(t, y + \mathbb{E}[P_t]) + \sqrt{1 - e^{-2a}} z \right) \phi(z) dz da. \end{aligned} \quad (5.5)$$

Step 2: Upper bound for g_{P_t}

Recall that for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$, which may vary from line to line, such that:

$$0 < v'(t, x) \leq C(\varepsilon) \left(1 + |x|^{\overline{\alpha}_{v'} + \varepsilon}\right), \quad \forall x \in \mathbb{R}.$$

Then, using (5.5) we get:

$$\begin{aligned} g_{P_t}(y) &\leq C(\varepsilon)t \left(1 + \left| v^{(-1)}(t, y + \mathbb{E}[P_t]) \right|^{\overline{\alpha}_{v'} + \varepsilon}\right) \\ &\quad \times \int_0^{+\infty} e^{-a} \int_{\mathbb{R}} \left(1 + \left| e^{-a} v^{(-1)}(t, y + \mathbb{E}[P_t]) + \sqrt{1 - e^{-2a}} z \right|^{\overline{\alpha}_{v'} + \varepsilon}\right) \phi(z) dz da \\ &\leq C(\varepsilon)t \left(1 + |v^{(-1)}(t, y + \mathbb{E}[P_t])|^{2(\overline{\alpha}_{v'} + \varepsilon)}\right). \end{aligned}$$

²Knowing that $D_r P_t$ does not depend on r , $\Phi_{P_t}(W) : [0, T] \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a random process which is actually constant on $[0, t]$.

By Lemma 5.2, $\overline{\alpha_{v(-1)}}$ belongs to $(0, +\infty)$, hence by the definition of $\overline{\alpha_{v(-1)}}$ it holds for all $\varepsilon' > 0$ that

$$g_{P_t}(y) \leq M'(\varepsilon, \varepsilon') t \left(1 + |y + \mathbb{E}[P_t]|^{2(\overline{\alpha_{v'}} + \varepsilon)(\overline{\alpha_{v(-1)}} + \varepsilon')} \right). \quad (5.6)$$

Step 3: Lower bound for g_{P_t}

Using Assumption (5.2) and (5.5) we have that

$$\begin{aligned} g_{P_t}(y) &\geq \frac{K^2 t}{1 + |v^{(-1)}(t, y + \mathbb{E}[P_t])|^{\tilde{\alpha}}} \\ &\quad \times \int_0^{+\infty} e^{-a} \int_{\mathbb{R}} \frac{1}{1 + |e^{-a}(v)^{-1}(t, y + \mathbb{E}[P_t])|^{\tilde{\alpha}} + |\sqrt{1 - e^{-2a}}z|^{\tilde{\alpha}}} \phi(z) dz da. \end{aligned}$$

Noticing that $|\sqrt{1 - e^{-2a}}z|^{\tilde{\alpha}} \leq |z|^{\tilde{\alpha}}$, and that there exists $C(\tilde{\alpha}) > 0$ such that

$$\int_{\mathbb{R}} \frac{(1 + |x|^{\tilde{\alpha}})\phi(z)}{1 + |x|^{\tilde{\alpha}} + |z|^{\tilde{\alpha}}} dz \geq C, \quad \forall x \in \mathbb{R}$$

we deduce that:

$$g_{P_t}(y) \geq \frac{CK^2 t}{1 + |v^{(-1)}(t, y + \mathbb{E}[P_t])|^{\tilde{\alpha}}} \int_0^{+\infty} e^{-a} \frac{1}{1 + e^{-a\tilde{\alpha}}|v^{(-1)}(t, y + \mathbb{E}[P_t])|^{\tilde{\alpha}}} da.$$

Hence: $g_{P_t}(y) \geq \frac{C'(\tilde{\alpha})t}{1 + |v^{(-1)}(t, y + \mathbb{E}[P_t])|^{2\tilde{\alpha}}}$. Since we know that for all $\varepsilon' > 0$, there exists a constant $\mu(\varepsilon') > 0$ such that for all $x \in \mathbb{R}$:

$$|v^{(-1)}(t, x)| \leq \mu(\varepsilon') \left(1 + |x|^{\overline{\alpha_{v(-1)}} + \varepsilon'} \right),$$

we finally get Relation (5.4) for some $M(\varepsilon') > 0$. We conclude using Nourdin and Viens' formula. \square

Corollary 5.5. *Let the assumptions in Theorem 5.4 hold, with the same notations. Assume moreover that $0 \leq \overline{\alpha_{v'}} < \underline{\alpha_v} < +\infty$. Then there exist $\varepsilon_0, \varepsilon'_0 > 0$, $y_0 > 0$ and $\gamma \in (0, 1)$ such that for any $|y| > y_0$:*

$$\rho_t(y) \leq \frac{\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{2M(\varepsilon'_0)t} \left(1 + |y|^{2\tilde{\alpha}(\overline{\alpha_{v(-1)}} + \varepsilon'_0)} \right) \exp \left(- \frac{|y - \mathbb{E}[P_t]|^{2(1-\gamma)} - |y_0 - \mathbb{E}[P_t]|^{2(1-\gamma)}}{4(1-\gamma)tM'(\varepsilon_0, \varepsilon'_0)} \right), \quad (5.7)$$

and

$$\begin{aligned} \rho_t(y) &\geq \frac{\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{2M'(\varepsilon_0, \varepsilon'_0)t(1 + |y|^\gamma)} \exp \left(- \frac{|y - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_{v(-1)}} + \varepsilon'_0) + 1)} - |y_0 - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_{v(-1)}} + \varepsilon'_0) + 1)}}{M(\varepsilon'_0)t(\tilde{\alpha}(\overline{\alpha_{v(-1)}} + \varepsilon'_0) + 1)} \right) \\ &\quad \times \exp \left(- \frac{|y_0 - \mathbb{E}[P_t]|^2}{M(\varepsilon'_0)t} \left(1 + y_0^{2\tilde{\alpha}(\overline{\alpha_{v(-1)}} + \varepsilon'_0)} \right) \right). \end{aligned} \quad (5.8)$$

Proof. Let us define for any $\varepsilon, \varepsilon' > 0$

$$\gamma(\varepsilon, \varepsilon') := (\overline{\alpha_{v'}} + \varepsilon)(\overline{\alpha_{v(-1)}} + \varepsilon').$$

Since we assumed that $0 \leq \overline{\alpha_{v'}} < \underline{\alpha_v} < +\infty$, we can deduce using Lemma 5.2 that there exist some $\varepsilon_0, \varepsilon'_0 > 0$ such that

$$\gamma := \gamma(\varepsilon_0, \varepsilon'_0) < 1.$$

We start with (5.7). We have from Theorem 5.4

$$\rho_t(y) \leq \frac{\mathbb{E}[|P_t - \mathbb{E}[P_t]|]}{2M(\varepsilon'_0)t} \left(1 + |y|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) \exp\left(-\int_0^{y-\mathbb{E}[P_t]} \frac{xdx}{M'(\varepsilon_0, \varepsilon'_0)t(1 + |x + \mathbb{E}[P_t]|^{2\gamma})}\right).$$

We notice that

$$\lim_{|x| \rightarrow +\infty} \frac{x}{M't(1 + |x + \mathbb{E}[P_t]|^{2\gamma})} \times \frac{1}{\frac{x}{M't|x|^{2\gamma}}} = 1,$$

so that there exists x_0 large enough such that $\frac{x}{M't(1 + |x + \mathbb{E}[P_t]|^{2\gamma})} \geq \frac{x}{2M't|x|^{2\gamma}}$ when $|x| \geq x_0$. Hence, since $\gamma \in (0, 1)$, we know that we can find some $y_0 > 0$ large enough such that if $|y| > y_0$

$$\begin{aligned} \int_{y_0 - \mathbb{E}[P_t]}^{y - \mathbb{E}[P_t]} \frac{xdx}{M't(1 + |x + \mathbb{E}[P_t]|^{2\gamma})} &\geq \int_{y_0 - \mathbb{E}[P_t]}^{y - \mathbb{E}[P_t]} \frac{xdx}{2M't|x|^{2\gamma}} \\ &= \frac{1}{4(1 - \gamma)tM'(\varepsilon_0, \varepsilon'_0)} \left(|y - \mathbb{E}[P_t]|^{2(1-\gamma)} - |y_0 - \mathbb{E}[P_t]|^{2(1-\gamma)}\right), \end{aligned}$$

from which (5.7) follows directly. Similarly, increasing y_0 if necessary, we have that for $|y| > y_0$

$$\begin{aligned} &\int_0^{y - \mathbb{E}[P_t]} x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) dx \\ &= \underbrace{\int_0^{y_0 - \mathbb{E}[P_t]} x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) dx}_{:=I_1} + \underbrace{\int_{y_0 - \mathbb{E}[P_t]}^{y - \mathbb{E}[P_t]} x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) dx}_{:=I_2}. \end{aligned}$$

Using the fact that the function $x \mapsto 1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}$ is convex, we deduce that for y_0 large enough

$$I_1 \leq |y_0 - \mathbb{E}[P_t]|^2 \left(1 + y_0^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right).$$

Moreover, since $\lim_{x \rightarrow +\infty} x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) \times \frac{1}{x^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1}} = 1$, we obtain for x large enough

$$x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) \leq 2x^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1}.$$

Then, we have that for $|y| \geq y_0$

$$I_2 \leq \frac{|y - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1)} - |y_0 - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1)}}{\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1}.$$

Hence,

$$\begin{aligned} &\int_0^{y - \mathbb{E}[P_t]} x \left(1 + |x + \mathbb{E}[P_t]|^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) dx \\ &\leq |y_0 - \mathbb{E}[P_t]|^2 \left(1 + y_0^{2\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0)}\right) + \frac{|y - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1)} - |y_0 - \mathbb{E}[P_t]|^{2(\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1)}}{\tilde{\alpha}(\overline{\alpha_v(-1)} + \varepsilon'_0) + 1}, \end{aligned}$$

from which the second inequality (5.8) follows directly using (5.4). \square

Finally, we have the following theorem, which is a simple application of the results obtained above in the special cases where we take the random variables (Y_t, Z_t) solutions to the BSDE (2.1) when they can be written $Y_t = v(t, W_t)$ and $Z_t = v'(t, W_t)$.

Theorem 5.6. *Let (Y, Z) be the solution to the BSDE (2.1) (which is assumed to exist and to be unique). Assume that there exists a map $v \in \mathcal{C}^{1,2}$ such that $Y_t = v(t, W_t)$.*

- (i) *If in addition, $v' > 0$, $0 \leq \overline{\alpha_{v'}} < \underline{\alpha_v} < +\infty$ and there exist $K > 0$, $\tilde{\alpha} > 0$ such that $v'(t, x) \geq 1/(K(1 + |x|^{\tilde{\alpha}}))$ then, denoting ρ_{Y_t} the density of the law of Y_t , there exists $y_0 > 0$, $C_1, C_2 > 0$, $p_1 \in (0, 2)$ and $p_2 > 0$ such that for any $|y| > y_0$*

$$\begin{aligned} \rho_{Y_t}(y) &\geq \frac{\mathbb{E}[|Y_t - \mathbb{E}[Y_t]|]}{C_2 t (1 + |y|^{1-p_1/2})} \exp \left(-\frac{|y - \mathbb{E}[Y_t]|^{2(p_2+1)} - |y_0 - \mathbb{E}[Y_t]|^{2(p_2+1)}}{(p_2 + 1)C_2 t} \right) \\ \rho_{Y_t}(y) &\leq \frac{\mathbb{E}[|Y_t - \mathbb{E}[Y_t]|]}{C_1 t} (1 + |y|^{2p_2}) \exp \left(-\frac{2|y_0 - \mathbb{E}[Y_t]|^2}{C_2 t} (1 + y_0^{2p_2}) \right) \\ &\quad \times \exp \left(-\frac{|y - \mathbb{E}[Y_t]|^{p_1} - |y_0 - \mathbb{E}[Y_t]|^{p_1}}{p_1 C_2 t} \right). \end{aligned}$$

- (ii) *If in addition, $v'' > 0$, $0 \leq \overline{\alpha_{v''}} < \underline{\alpha_{v'}} < +\infty$ and there exist $K > 0$, $\tilde{\alpha} > 0$ such that $v''(t, x) \geq 1/(K(1 + |x|^{\tilde{\alpha}}))$ then, denoting ρ_{Z_t} the density of the law of Z_t , there exists $z_0 > 0$, $C_1, C_2 > 0$, $p_1 \in (0, 2)$ and $p_2 > 0$ such that for any $|z| > z_0$*

$$\begin{aligned} \rho_{Z_t}(z) &\geq \frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{C_2 t (1 + |z|^{1-p_1/2})} \exp \left(-\frac{|z - \mathbb{E}[Z_t]|^{2(p_2+1)} - |z_0 - \mathbb{E}[Z_t]|^{2(p_2+1)}}{(p_2 + 1)C_2 t} \right) \\ \rho_{Z_t}(z) &\leq \frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{C_1 t} (1 + |z|^{2p_2}) \exp \left(-\frac{2|z_0 - \mathbb{E}[Z_t]|^2}{C_2 t} (1 + z_0^{2p_2}) \right) \\ &\quad \times \exp \left(-\frac{|z - \mathbb{E}[Z_t]|^{p_1} - |z_0 - \mathbb{E}[Z_t]|^{p_1}}{p_1 C_2 t} \right). \end{aligned}$$

5.3 Verifying the assumptions of Theorem 5.6

In this subsection, we give some conditions which ensure that the assumptions in Corollary 5.5 hold. We recall that under Assumptions (A)(i), (L) or (Q), (D1) and according to Proposition 4.1, there exists a map $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ in $\mathcal{C}^{1,2}$ such that $Y_t = u(t, W_t)$, $t \in [0, T]$, \mathbb{P} -a.s., and Z admits a continuous version given by $Z_t = u'(t, W_t)$, $t \in [0, T]$, \mathbb{P} -a.s., assuming that $\sigma \equiv 1$ and $b \equiv 0$ in the studied FBSDE (2.1). Moreover we suppose for simplicity that the generator h of BSDE (2.1) depends only on z , and that u' and u'' are³ in $\mathcal{C}^{1,2}$. By a simple application of the non-linear Feynman-Kac formula (see for instance [24]), and by differentiating it repeatedly, it can be shown that u , u' and u'' are respectively classical solutions of the following PDEs:

$$\begin{cases} -u_t(t, x) - \frac{1}{2}u_{xx}(t, x) - h(t, u_x(t, x)) = 0, & (t, x) \in [0, T] \times \mathbb{R} \\ u(T, x) = g(x), & x \in \mathbb{R}, \end{cases} \quad (5.9)$$

$$\begin{cases} -u'_t(t, x) - \frac{1}{2}u'_{xx}(t, x) - h_z(t, u'(t, x))u'_x(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R} \\ u'(T, x) = g'(x), & x \in \mathbb{R}, \end{cases} \quad (5.10)$$

$$\begin{cases} -u''_t(t, x) - \frac{1}{2}u''_{xx}(t, x) - h_z(t, u'(t, x))u''_x(t, x) - h_{zz}(t, u'(t, x))|u''(t, x)|^2 = 0, & (t, x) \in [0, T] \times \mathbb{R} \\ u''(T, x) = g''(x), & x \in \mathbb{R}. \end{cases} \quad (5.11)$$

³This assumption is satisfied if g and h are smooth enough.

We show in the following proposition and its corollary that under some conditions on g, g', g'' and h, h_z , the assumptions in Theorem 5.6 are satisfied. We emphasize that this is only one possible set of assumptions, and that the required properties of u and its derivatives can be checked on a case by case analysis.

Proposition 5.7. *Let u, u' and u'' be respectively the solution to (5.9), (5.10) and (5.11) and assume that a comparison theorem holds for classical super and sub-solutions of these PDEs, in the class of functions with polynomial growth. Assume that there exist $(\varepsilon, \underline{C}, \overline{C}) \in (0, 1) \times (0, +\infty)^3$, such that for all $x \in \mathbb{R}$*

$$\underline{C}(1 + |x|^{1-\varepsilon}) \leq g(x) \leq \overline{C}(1 + |x|^{1+\varepsilon}).$$

Assume moreover that h is non-positive and that there exist $(\varepsilon', \underline{D}, \overline{D}) \in (0, \varepsilon) \times (0, +\infty)^2$ s.t.

$$\underline{D}(1 + |x|^{\varepsilon'}) \leq g'(x) \leq \overline{D}(1 + |x|^{\varepsilon}).$$

Assume that there exist $(\underline{B}, \overline{B}) \in (0, +\infty)^2$ such that for all $x \in \mathbb{R}$

$$\underline{B} \leq g''(x) \leq \overline{B}, \text{ and } 0 \leq h_{zz}(t, x) < \frac{1}{4\overline{B}T}.$$

Assume finally that there exist $\lambda \in (0, \varepsilon^{-1} - 1]$ and $C > 0$ such that $|h_z(t, z)| \leq C(1 + |z|^\lambda)$, then for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$\underline{\alpha}_u \in [1 - \varepsilon, 1 + \varepsilon], \quad \overline{\alpha}_{u'}, \underline{\alpha}_{u'} \in [\varepsilon', \varepsilon], \quad \overline{\alpha}_{u''} = 0, \quad u'(t, x) \geq \underline{D} \text{ and } u''(t, x) \geq \underline{B}.$$

Proof. Let $\varphi(t, x) := \tilde{C}(T - t) + \overline{C}k_\varepsilon(x)$, where $k_\varepsilon(x)$ is in $\mathcal{C}^\infty(\mathbb{R})$, coincides with the function $(1 + |x|^{1+\varepsilon})$ outside some closed interval centered at 0 and is always greater than $(1 + |x|^{1+\varepsilon})$. We show that φ is a (classical) super-solution to (5.9) for some positive constant \tilde{C} large enough. Indeed we can choose $\tilde{C} > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}$

$$-\varphi_t(t, x) - \frac{1}{2}\varphi_{xx}(t, x) - h(t, \varphi_x(t, x)) = \tilde{C} - \frac{1}{2}\overline{C}k_\varepsilon''(x) - h(t, \varphi_x(t, x)) \geq 0,$$

since $h \leq 0$ and $\lim_{|x| \rightarrow \infty} \frac{1}{2}k_\varepsilon''(x) = 0$.

Moreover, by the assumption made on g , we clearly have for all $x \in \mathbb{R}$, $g(x) \leq \overline{C}k_\varepsilon(x)$, so that we deduce by comparison that for all $(t, x) \in [0, T] \times \mathbb{R}$:

$$u(t, x) \leq \overline{C}k_\varepsilon(x) + \tilde{C}(T - t).$$

Now, we let $\phi(t, x) := -\tilde{C}_1(T - t) + \underline{C}\kappa_\varepsilon(x)$ for $(t, x) \in [0, T] \times \mathbb{R}$, where $\kappa_\varepsilon(x)$ is in $\mathcal{C}^\infty(\mathbb{R})$, coincides with the function $(1 + |x|^{1-\varepsilon})$ outside some closed interval centered at 0 and is always smaller than $(1 + |x|^{1-\varepsilon})$. We show that ϕ is a classical subsolution to (5.9) for some positive constant \tilde{C}_1 large enough. We have

$$-\phi_t(t, x) - \frac{1}{2}\phi_{xx}(t, x) - h(t, \phi_x(t, x)) = -\tilde{C}_1 + \frac{1}{2}\underline{C}\kappa_\varepsilon''(x) - h(t, \phi_x(t, x)). \quad (5.12)$$

Given that the quantity $h(t, \phi_x(t, x)) = h(t, \underline{C}\kappa_\varepsilon'(x))$ is bounded because $\lim_{|x| \rightarrow \infty} \kappa_\varepsilon'(x) = 0$ and h is continuous, we can always choose \tilde{C}_1 so that (5.12) is non-positive. Then, since we clearly have for all $x \in \mathbb{R}$, $g(x) \geq \underline{C}\kappa_\varepsilon(x)$, we deduce by comparison that for all $(t, x) \in [0, T] \times \mathbb{R}$:

$$u(t, x) \geq \overline{C}\kappa_\varepsilon(x) + \tilde{C}_1(T - t).$$

To sum up, we have showed that for all $(t, x) \in [0, T] \times \mathbb{R}$:

$$\underline{C}\kappa_\varepsilon(x) - \tilde{C}_1(T - t) \leq u(t, x) \leq \overline{C}\kappa_\varepsilon(x) + \tilde{C}(T - t).$$

In other words $[\alpha_u, \overline{\alpha}_u] \subset [1 - \varepsilon, 1 + \varepsilon]$.

We now study (5.10). Define for some constant $\tilde{C}_2 > 0$ to be fixed later

$$\psi(t, x) := \tilde{C}_2(T - t) + \overline{D}\Upsilon_\varepsilon(x),$$

where $\Upsilon_\varepsilon(x)$ is in $\mathcal{C}^\infty(\mathbb{R})$, coincides with the function $(1 + |x|^\varepsilon)$ outside some closed interval centered at 0 and is always greater than $(1 + |x|^\varepsilon)$. We then have

$$-\psi_t(t, x) - \frac{1}{2}\psi_{xx}(t, x) - h_z(t, \psi(t, x))\psi_x(t, x) = \tilde{C}_2 - \frac{1}{2}\overline{D}\Upsilon_\varepsilon''(x) - h_z(t, \psi(t, x))\overline{D}\Upsilon_\varepsilon'(x).$$

Next, for some constant $C > 0$ which may vary from line to line

$$|h_z(t, \psi(t, x))| \leq C(1 + |\psi(t, x)|^\lambda) \leq C(1 + |x|^{\lambda\varepsilon}),$$

and since $\lambda \leq \frac{1}{\varepsilon} - 1$ we deduce that:

$$|h_z(t, \psi(t, x))\overline{D}\Upsilon_\varepsilon'(x)| \leq C(1 + |x|^{\lambda\varepsilon + \varepsilon - 1}), \text{ which is bounded.}$$

Since in addition we have $\Upsilon_\varepsilon''(x) \rightarrow 0$ as $|x|$ goes to $+\infty$, we can always choose \tilde{C}_2 large enough so that

$$-\psi_t(t, x) - \frac{1}{2}\psi_{xx}(t, x) - h_z(t, \psi(t, x))\psi_x(t, x) \geq 0.$$

By the assumption we made on g , we can use once more the comparison theorem to obtain

$$u'(t, x) \leq \psi(t, x).$$

Similarly, we show that $\underline{D}\Upsilon_{\varepsilon'}(x) - \tilde{C}_3(T - t)$ is a sub-solution of (5.10) for some positive constant \tilde{C}_3 , since $\lambda \leq \varepsilon^{-1} - 1 \leq \varepsilon'^{-1} - 1$. Then, by comparison, we deduce that $\overline{\alpha}_{u'}, \underline{\alpha}_{u'} \in [\varepsilon', \varepsilon]$. Moreover, we notice that $\underline{D} \leq g'(x)$ for all $x \in \mathbb{R}$, so \underline{D} is a sub-solution of (5.10). Thus, using once more the comparison theorem $u'(t, x) \geq \underline{D}$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

We now study (5.11). Given that h_{zz} is non negative and $\underline{B} \leq g''(x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$, we deduce directly that \underline{B} is a sub-solution of (5.11). Next, let $\varpi(t, x) = \overline{B} + \frac{\overline{B}}{T^{1-\eta}}(T - t)^{1-\eta}$ where $\eta \in (0, 1)$ is chosen small enough so that $h_{zz}(t, x) \leq \frac{1-\eta}{4T\overline{B}}$. Thus,

$$\begin{aligned} & -\varpi_t(t, x) - \frac{1}{2}\varpi_{xx}(t, x) - h_z(t, u'(t, x))\varpi_x(t, x) - h_{zz}(t, u'(t, x))|\varpi(t, x)|^2 \\ &= (1 - \eta)\frac{\overline{B}}{T^{1-\eta}}(T - t)^{-\eta} - h_{zz}(t, u'(t, x))\overline{B}^2 \left(1 + \frac{(T - t)^{1-\eta}}{T^{1-\eta}}\right)^2 \\ &\geq (1 - \eta)\frac{\overline{B}}{T^{1-\eta}}(T - t)^{-\eta} - \frac{1 - \eta}{4T}\overline{B} \left(1 + \frac{(T - t)^{1-\eta}}{T^{1-\eta}}\right)^2 \\ &\geq 0. \end{aligned}$$

We deduce that ϖ is a super solution of (5.11), which by comparison, implies that u'' is bounded, so $\overline{\alpha}_{u''} = 0$.

Corollary 5.8. *Consider the FBSDE (2.1) and assume that for all $t \in [0, T]$ $X_t = W_t$ and h depends only on z . Let $u(t, X_t) := Y_t$ and assume that $u \in \mathcal{C}^{1,2}$, $u' \in \mathcal{C}^{1,2}$ and $u'' \in \mathcal{C}^{1,2}$. Let the assumptions of Proposition 5.7 hold, and assume moreover that $\varepsilon \in (0, \frac{1}{2})$. Then, the assumptions of Theorem 5.6 hold.*

Proof. According to Proposition 5.7, $\underline{\alpha}_u \geq 1 - \varepsilon$, $\overline{\alpha}_{u'} \leq \varepsilon$ and $u'(t, x) \geq \underline{D}$, $(t, x) \in [0, T] \times \mathbb{R}$. From the fact that ε is smaller than $1/2$, we deduce that $0 \leq \overline{\alpha}_{u'} < \underline{\alpha}_u < +\infty$. Moreover, $0 = \overline{\alpha}_{u''} < \varepsilon' \leq \underline{\alpha}_{u'}$. \square

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References

- [1] O. Aboura and S. Bourguin. Density Estimates for Solutions to One Dimensional Backward SDE's. *Potential Anal.*, 38(2):573–587, 2013.
- [2] S. Ankirchner, P. Imkeller, and G. Dos Reis. Classical and variational differentiability of BSDEs with quadratic growth. *Electron. J. Probab.*, 12:no. 53, 1418–1453 (electronic), 2007.
- [3] F. Antonelli and A. Kohatsu-Higa. Densities of one-dimensional backward SDEs. *Potential Anal.*, 22(3):263–287, 2005.
- [4] N. H. Bingham, C.M. Goldie, and J. L. Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.
- [5] J.-F. Chassagneux and A. Richou. Numerical simulation of quadratic BSDEs. Preprint. <http://arxiv.org/abs/1307.5741>, 2013.
- [6] G. Dos Reis. *On some properties of solutions of quadratic growth BSDE and applications in finance and insurance*. PhD thesis, Humboldt University in Berlin, <http://www.math.tu-berlin.de/~dosreis/publications/GdosReis-PhD-Thesis.pdf>, 2010.
- [7] G. dos Reis and R. J. N. dos Reis. A note on comonotonicity and positivity of the control components of decoupled quadratic FBSDE. *Stoch. Dyn.*, 13(4):1350005, 11, 2013.
- [8] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 13(2):99–125, 1977.
- [9] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.
- [10] N. Fournier and J. Printems. Absolute continuity for some one-dimensional processes. *Bernoulli*, 16(2):343–360, 2010.
- [11] P. Imkeller and G. Dos Reis. Path regularity and explicit convergence rate for BSDE with truncated quadratic growth. *Stochastic Process. Appl.*, 120(3):348–379, 2010.

- [12] P. Imkeller, A. Réveillac, and A. Richter. Differentiability of quadratic bsde generated by continuous martingales and hedging in incomplete markets. *Annals of Applied Probability*, 22(1):285–336, 2012.
- [13] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [14] N. Kazamaki. *Continuous exponential martingales and BMO*, volume 1579 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.
- [15] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, 28(2):558–602, 2000.
- [16] A. Kohatsu-Higa. Lower bounds for densities of uniformly elliptic non-homogeneous diffusions. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 323–338. Birkhäuser, Basel, 2003.
- [17] A. Kohatsu-Higa. Lower bounds for densities of uniformly elliptic random variables on Wiener space. *Probab. Theory Related Fields*, 126(3):421–457, 2003.
- [18] J. Ma and J. Zhang. Path regularity for solutions of backward stochastic differential equations. *Probab. Theory Related Fields*, 122(2):163–190, 2002.
- [19] J. Ma and J. Zhang. Representation theorems for backward stochastic differential equations. *Ann. Appl. Probab.*, 12(4):1390–1418, 2002.
- [20] I. Nourdin and F. Viens. Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Probab.*, 14:no. 78, 2287–2309, 2009.
- [21] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
- [22] E. Nualart and L. Quer-Sardanyons. Gaussian estimates for the density of the non-linear stochastic heat equation in any space dimension. *Stochastic Process. Appl.*, 122(1):418–447, 2012.
- [23] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [24] E. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, 1992.
- [25] E. Seneta. *Regularly varying functions*. Lecture Notes in Mathematics, Vol. 508. Springer-Verlag, Berlin, 1976.